Lecture 22: The Heron-Rota-Welsh conjecture

(joint w/ Petter Brändén)

Alternative description of a matroid: lattice of flats, $\mathcal{L}(M)$.

(If M is representable by $\{v_1, \dots, v_n\} \subseteq \mathbb{F}^d$, then the flats of M are all sets obtained by intersecting $\{v_1, \dots, v_n\}$ with any subspace of $\mathbb{F}^d$.)

The <u>characteristic polynomial</u> of M is:

$$\chi_M(t) = \sum_{F \in \mathcal{L}(M)} \mu(\hat{0}, F) \cdot t^{rk(\hat{1}) - rk(F)},$$

where μ is the Möbius function.

(We will not define μ formally.)

The <u>reduced char. polyn.</u> of M is:

$$\overline{\chi}_M(t) = \frac{\chi_M(t)}{t-1} \quad (\text{when } rk(M) \geq 1).$$

<u>Lemma</u>: If i is not a loop,

$$\overline{\chi}_M(t) = \sum_{F \not\ni i} \mu(\hat{0}, F) \cdot t^{rk(\hat{1}) - rk(F) - 1}.$$

<u>Goal</u>: Show the absolute values of the coefficients of $\overline{\chi}_M(t)$ form a log-concave sequence.

<u>Definition</u>: Given an open convex cone $C \subseteq \mathbb{R}^n$ and d-homog. polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$, we say p is C-Lorentzian if:

(P) $\forall v_1, \dots, v_d \in C$, $D_{v_1} \cdots D_{v_d} p > 0$

(Q) $\forall v_1, \dots, v_{d-2} \in C$, $D_{v_1} \cdots D_{v_{d-2}} p$ has associated matrix with exactly one positive eigenvalue.

<u>Lemma</u>: Given C-Lorentzian p and $v, w \in \overline{C}$, $f(t,s) = p(t \cdot v + s \cdot w)$ has Ultra log-concave coefficients:

$$f(t,s)=\sum_{k=0}^{d}\binom{d}{k}c_k t^k s^{d-k}$$

and $c_k^2 \geq c_{k-1}\cdot c_{k+1}$ $\forall k$.

Proof: Local 3-term inequalities follow from 1 pos. eval. condition.

Strategy:

Goal $\Leftrightarrow$

$$f(t,s)=\sum_{F\ni i}\binom{rk(\mathbb{1})-1}{rk(F)}|\mu(\hat{0},F)|\,s^{rk(F)}t^{rk(\mathbb{1})-rk(F)-1}$$

has ultra log-concave coeff.

(Mobius func. alternates sign)

1) Construct C-Lorentzian polynomial p

2) Choose $v,w\in\mathbb{R}^n_{\geq 0}$ s.t.

$$f(t,s)=p(t\cdot v+s\cdot w).$$

Candidate polynomial

Let E be the ground set of M.

Given $K<L$, flats in $\mathcal{L}(M)$, define:

$$\mathcal{E}_K^L := \{(y_S)_{K \subset S \subset L} : y_S \in \mathbb{R}\} \cong \mathbb{R}^{2^{|L \setminus K|}-2}$$

$$\mathcal{M}_K^L := \{ \bar{y} \in \mathcal{E}_K^L : \\ y_S + y_T = y_{S \cap T} + y_{S \cup T} \quad \forall S, T \\ \text{where } y_K, y_L := 0 \}$$

(This is the subspace of <u>modular</u> set functions w/ zero endpoints.)

$$\mathcal{C}_K^L := \{ \bar{y} \in \mathcal{E}_K^L : \\ y_S + y_T > y_{S \cap T} + y_{S \cup T} \quad \forall S, T \text{ incomparable} \\ \text{where } y_K, y_L := 0 \}.$$

(This is the open convex cone of all <u>strictly submodular</u> set functions w/ zero endpoints.)

<u>Note</u>: $\bar{y} \in \mathcal{M}_K^L$ if and only if there are real $(c_e)_{e \in L \setminus K}$ s.t. $\sum_{e \in L \setminus K} c_e = 0$ and

$$y_S = \sum_{e \in S \setminus K} c_e \text{ for all } K \subset S \subset L.$$

Now, given $K \leq F < G \leq L$, all flats in $\mathcal{L}(M)$, define linear maps:

$\pi_F^G : \mathcal{E}_K^L \to \mathcal{E}_F^G$ via

$$\pi_F^G(\bar{t}) = \left(t_S - t_G \cdot \frac{|S \setminus F|}{|G \setminus F|} - t_F \cdot \frac{|G \setminus S|}{|G \setminus F|} \right)_{F \subset S \subset G}$$

where $t_K, t_L := 0$.

(Note that π_F^G simply subtracts modular set functions from $\bar{t}$ to enforce $\pi_F^G(\bar{t})$ is 0 at the endpoints F and G.)

Thus, $\pi_F^G(\mathcal{M}_K^L) \subseteq \mathcal{M}_F^G$ and

$$\pi_F^G(\mathcal{C}_K^L) \subseteq \mathcal{C}_F^G.$$

__Definition__: Given $K < L$, flats in $\mathcal{L}(M)$, define $r(K,L) = rk(L) - rk(K)$ and $d(K,L) = r(K,L) - 1$. We define $pol_K^L(\bar{t})$ on $\mathcal{E}_K^L$ recursively as follows:

1) If $d(K,L) = 0$, then $pol_K^L(\bar{t}) = 1$,

2) If $d(K,L) \geq 1$, then

$$d(K,L) \cdot pol_K^L(\bar{t}) := \sum_{K < F < L} t_F \cdot pol_K^F(\pi_K^F(\bar{t})) \cdot pol_F^L(\pi_F^L(\bar{t})).$$

(<u>Notice</u>: $d(K,L) = \deg(\mathrm{pol}_K^L)$, and pol_K^L actually only depends on t_F, $F \in \mathcal{L}(M)$.)

<u>E.g.</u>: If $d(K,L) = 1 \Leftrightarrow r(K,L) = 2$, then

$$1 \cdot \mathrm{pol}_K^L(\bar{t}) = \sum_{K \prec F \prec L} t_F \cdot 1 \cdot 1 = \sum_{K \prec F \prec L} t_F .$$

<u>E.g.</u>: If $d(K,L) = 2 \Leftrightarrow r(K,L) = 3$, then

$$2 \cdot \mathrm{pol}_K^L(\bar{t}) = \sum_{K \prec F} t_F \cdot 1 \cdot \mathrm{pol}_F^L(\pi_F^L(\bar{t})) + \sum_{G \prec L} t_G \cdot \mathrm{pol}_K^G(\pi_K^G(\bar{t})) \cdot 1$$

$$\pi_F^L(\bar{t}) = \left(t_G - t_F \cdot \frac{|L \setminus G|}{|L \setminus F|}\right)_{F \prec G \prec L}$$

$$\pi_K^G(\bar{t}) = \left(t_F - t_G \cdot \frac{|F \setminus K|}{|G \setminus K|}\right)_{K \prec F \prec G}$$

$$= \sum_{K \prec F} t_F \cdot \sum_{F \prec G \prec L} \left(t_G - t_F \cdot \frac{|L \setminus G|}{|L \setminus K|}\right) + \sum_{G \prec L} t_G \sum_{K \prec F \prec G} \left(t_F - t_G \cdot \frac{|F \setminus K|}{|G \setminus K|}\right)$$

$$= \sum_{K \prec F \prec G \prec L} \left(2 t_F t_G - t_F^2 \cdot \frac{|L \setminus G|}{|L \setminus F|} - t_G^2 \cdot \frac{|F \setminus K|}{|G \setminus K|}\right)$$

<u>Step 1 Plan</u>: Prove that $\mathrm{pol}_K^L(\bar{t})$ is $\mathcal{C}_K^L$-Lorentzian by induction.

Properties of $\mathcal{E}_K^L, \mathcal{M}_K^L, \mathcal{C}_K^L$:

1) $\mathcal{M}_K^L$ is the linearity space of $\mathcal{C}_K^L$.
2) $\mathcal{C}_K^L$ is effective (i.e., given any strictly submodular $y \in \mathcal{C}_K^L$, there exists modular $w \in \mathcal{M}_K^L$ s.t. $y+w$ has strictly pos. entries)
3) If $K \subseteq F \subset G \subseteq L$, then
$$\pi_F^G(\mathcal{M}_K^L) \subseteq \mathcal{M}_F^G, \quad \pi_F^G(\mathcal{C}_K^L) \subseteq \mathcal{C}_F^G$$
4) If $K \subseteq A \subseteq F \subset G \subseteq B \subseteq L$, then
$$\pi_F^G(\pi_A^B(\bar{x})) = \pi_F^G(\bar{x})$$

Pf.: Fix S, $F \subset S \subset G$

$$\pi_A^B(x) = \left(x_S - x_B \cdot \frac{|S\setminus A|}{|B\setminus A|} - x_A \frac{|B\setminus S|}{|B\setminus A|} \right)_S =: \bar{x}'$$

$$\pi_F^G(\bar{x}') = \left(x'_S - x'_G \frac{|S\setminus F|}{|G\setminus F|} - x'_F \cdot \frac{|G\setminus S|}{|G\setminus F|} \right)_S$$

$$\hookrightarrow \left(x_S - x_B \cdot \frac{|S\setminus A|}{|B\setminus A|} - x_A \frac{|B\setminus S|}{|B\setminus A|} \right)$$
$$- \left(x_G - x_B \frac{|G\setminus A|}{|B\setminus A|} - x_A \frac{|B\setminus G|}{|B\setminus A|} \right) \cdot \frac{|S\setminus F|}{|G\setminus F|}$$
$$- \left(x_F - x_B \frac{|F\setminus A|}{|B\setminus A|} - x_A \frac{|B\setminus F|}{|B\setminus A|} \right) \cdot \frac{|G\setminus S|}{|G\setminus F|}$$

To show: $\dfrac{|G\setminus A|\cdot|S\setminus F| + |F\setminus A|\cdot|G\setminus S|}{|G\setminus F|} = |S\setminus A|$

(and similar for t_A)

$A = \left[(|G\setminus F| + |F\setminus A|)(|G\setminus F| - |G\setminus S|) + |F\setminus A|\cdot|G\setminus S|\right]/|G\setminus F|$

$= |G\setminus F| + |F\setminus A| - |G\setminus S| = |G\setminus A| - |G\setminus S| = |S\setminus A|$ ✓

$\square$

Properties of $\mathrm{pol}_K^L(\bar{t})$:

Lemma: S $f \in \mathbb{R}[t_1,\dots,t_n]$ is d-homogeneous, and $d\cdot f(\bar{t}) = \sum_{i=1}^{n} t_i \cdot Q_i(\bar{t})$, Q_i $(d-1)$-homog.. If $\partial_{t_i} Q_j = \partial_{t_j} Q_i$ $\forall i,j$, then $Q_i = \partial_{t_i} f$, $\forall i$.

Pf.: Euler's identity $\Rightarrow d\cdot\partial_{t_j} f$

$= Q_j + \sum_{i=1}^{n} t_i \partial_{t_j} Q_i = Q_j + \sum_{i=1}^{n} t_i \partial_{t_i} Q_j$

$= Q_j + (d-1)Q_j = d\cdot Q_j$ $\square$

Lemma: If $K < F < L$, then

$\partial_{t_F} \mathrm{pol}_K^L(\bar{t}) = \mathrm{pol}_K^F(\pi_K^F(\bar{t}))\cdot \mathrm{pol}_F^L(\pi_F^L(\bar{t}))$.

Pf.: We first prove if $F, G \in \mathcal{L}(M)$ are incomparable, then $\partial_{t_F}\partial_{t_G} \mathrm{pol}_K^L(\bar{t})$ by induction on $d = d(K,L)$.

First note that

$$\mathrm{pol}_K^F(\pi_K^F(\bar{t}))\cdot \mathrm{pol}_F^L(\pi_F^L(\bar{t}))$$

does not depend on t_G.

Thus, $\partial_{t_F}\partial_{t_G}[d\cdot \mathrm{pol}_K^L(\bar{t})]$

$$= \sum_{K<S<G} t_S \cdot \partial_{t_F}\partial_{t_G}\left[\mathrm{pol}_K^S(\pi_K^S(\bar{t}))\cdot \mathrm{pol}_S^L(\pi_S^L(\bar{t}))\right]$$

$= 0$, by induction.

(d=1 clear)

We now prove the claim by induction on d. Def $Q_F(\bar{t}) = \mathrm{pol}_K^F(\pi_K^F(\bar{t}))\cdot \mathrm{pol}_F^L(\pi_F^L(\bar{t}))$, for any $G<F$, we have

$$\partial_{t_G} Q_F(\bar{t}) = \mathrm{pol}_K^G(\pi_K^G(\pi_K^F(\bar{t}))) \cdot \mathrm{pol}_G^F(\pi_G^F(\pi_K^F(\bar{t})))\cdot \mathrm{pol}_F^L(\pi_F^L(\bar{t}))$$

$$= \mathrm{pol}_K^G(\pi_K^G(\bar{t}))\cdot \mathrm{pol}_G^F(\pi_G^F(\bar{t}))\cdot \mathrm{pol}_F^L(\pi_F^L(\bar{t})).$$

Thus, $\partial_{t_G} Q_F(t) = \partial_{t_F} Q_G(t)$, and the claim follows from the previous lemma.

<u>Lemma</u>: $\forall y \in \Sigma_K^L$, $\ell \in \mathcal{M}_K^L$,

$$\mathrm{pol}_K^L(y+\ell) = \mathrm{pol}_K^L(y)$$

$$\Longleftrightarrow D_\ell \, \mathrm{pol}_K^L(\bar{t}) \equiv 0 \quad \forall \ell \in \mathcal{M}_K^L.$$

Proof: By induction on $d = d(K, L)$.

If $d = 1$, we have

$$\mathrm{pol}_K^L(\ell) = \sum_{K \lessdot F \lessdot L} \ell_F \underset{\substack{\uparrow \\ \text{modular}}}{=} \sum_{K \lessdot F \lessdot L} \sum_{e \in F \setminus K} c_e$$

where $\sum_{e \in L \setminus K} c_e = 0$.

Thus, $\mathrm{pol}_K^L(\ell) = \sum_{e \in L \setminus K} \sum_{F \ni e} c_e$.

Since $F \setminus K$ partitions $L \setminus K$,

$$= \sum_{e \in L \setminus K} c_e = 0.$$

For $d > 1$, $D_\ell \mathrm{pol}_K^L(y) \cong D_\ell D_y \mathrm{pol}_K^L(y)$ by homogeneity, and by prev. lemma,

$$D_\ell D_y \mathrm{pol}_K^L(y) = D_\ell \Big|_{\bar{x} = y} \sum_{K < F < L} y_\ell \cdot \mathrm{pol}_K^F(\pi_K^F(\bar{x})) \cdot \mathrm{pol}_F^L(\pi_F^L(\bar{x}))$$

Product rule, induction, and $\pi_K^F(\mathcal{M}_K^L) \subseteq \mathcal{M}_K^F$ imply $D_\ell D_y \mathrm{pol}_K^L(y) = 0$.

Thus, $D_\ell \mathrm{pol}_K^L \equiv 0$. $\square$

<u>Lemma</u>: $\forall v_1, \dots, v_d \in \mathcal{C}_K^L$,

$D_{v_1} \cdots D_{v_d} \mathrm{pol}_K^L > 0$, and

$\nabla^2 [D_{v_1} \cdots D_{v_{d-2}} \mathrm{pol}_K^L]$ has non-neg. off-diag. entries.

<u>Proof</u>: By induction on d.

By previous Lemma and effectiveness of $\mathcal{C}_K^L$, we may assume that v_d has strictly positive entries. Thus,

$D_{v_1} \cdots D_{v_d} \mathrm{pol}_K^L(\bar{x}) =$

$$\sum_{K<F<L} (v_d)_F \, D_{v_1} \cdots D_{v_{d-1}} [\mathrm{pol}_K^F(\pi_K^F(\bar{x})) \cdot \mathrm{pol}_F^L(\pi_F^L(\bar{x}))]$$

Note that $D_v \mathrm{pol}_K^F(\pi_K^F(\bar{x}))$

$$= [D_{\pi_K^F(v)} \mathrm{pol}_K^F](\pi_K^F(\bar{x})).$$

Since $\pi_K^F(\mathcal{C}_K^L) \subseteq \mathcal{C}_K^F$, induction implies $D_{v_1} \cdots D_{v_d} \mathrm{pol}_K^L(\bar{x})$ is a sum of positive numbers. For the Hessian, $K<F<G<L$ implies:

$$\partial_{x_F} \partial_{x_G} \mathrm{pol}_K^L(\bar{x}) = \mathrm{pol}_K^F(\pi_K^F(\bar{x})) \cdot \mathrm{pol}_F^G(\pi_F^G(\bar{x})) \cdot \mathrm{pol}_G^L(\pi_G^L(\bar{x}))$$

and the same argument works.

If F, G incomparable, then

$\partial_{t_F}\partial_{t_G}\,\mathrm{pol}_K^L(\bar{t}) \equiv 0$, as above. $\square$

<u>Next step</u>: Prove that

$\mathrm{pol}_K^L(\bar{t})$ is $\mathcal{C}_K^L$-Lorentzian.

Lecture 23: Heron-Rota-Welsh Continued. [SIDE BOARD]

Last time: Matroid M with lattice of flats $\mathcal{L}(M)$. Collection of polynomials pol_K^L for $K<L$ in $\mathcal{L}(M)$:

$d(K,L)=0 \to \mathrm{pol}_K^L(t)=1$

$d(K,L)\geq 1 \to$

$$d(K,L)\cdot \mathrm{pol}_K^L(t) = \sum_{K<F<L} t_F \cdot \mathrm{pol}_K^F(\pi_K^F(t))\cdot \mathrm{pol}_F^L(\pi_F^L(t))$$

polynomials defined on

$$\Sigma_K^L := \{(y_S)_{K\subseteq S\subseteq L} : y_S \in \mathbb{R}\}$$

and to prove $\mathcal{C}_K^L$-Lorentzian where

$$\mathcal{C}_K^L := \{y\in \Sigma_K^L :$$
$$y_S + y_T > y_{S\cap T} + y_{S\cup T},\ S,T \text{ incomparable}$$
$$y_K = y_L = 0\}$$ is open, convex cone

with lineality space

$$\mathcal{M}_K^L := \{y\in\Sigma_K^L :$$
$$y_S + y_T = y_{S\cap T} + y_{S\cup T} \quad \forall S,T,$$
$$y_K = y_L = 0\}.$$

Properties of $\mathrm{pol}_K^L(t)$ and π_K^L:

1) $\partial_{t_F} \mathrm{pol}_K^L(t) = \begin{cases} 0, & \text{if not } K<F<L \\ \mathrm{pol}_K^F(\pi_K^F(t)) \cdot \mathrm{pol}_F^L(\pi_F^L(t)) & \text{otherwise} \end{cases}$

2) If $K \subseteq F < G \subseteq L$, then
$\pi_F^G(\mathcal{C}_K^L) \subseteq \mathcal{C}_F^G$ and $\pi_F^G(\mathcal{M}_K^L) \subseteq \mathcal{M}_F^G$

3) If $K \leq A \subseteq F < G \subseteq B \subseteq L$, then
$(\pi_F^G \circ \pi_A^B)(t) = \pi_F^G(t) \quad \forall t \in \mathcal{E}_K^L$

4) $\mathrm{pol}_K^L(t+w) = \mathrm{pol}_K^L(t) \quad \forall w \in \mathcal{M}_K^L$

5) $\forall y \in \mathcal{C}_K^L \; \exists w \in \mathcal{M}_K^L$ s.t. $y+w$ has strictly pos. entries

6) $D_{v_1} \cdots D_{v_d} \mathrm{pol}_K^L(t) > 0 \quad \forall v_1, \dots, v_d \in \mathcal{C}_K^L$

7) $\partial_{t_F} \partial_{t_G} D_{v_1} \cdots D_{v_{d-2}} \mathrm{pol}_K^L(t) \geq 0 \quad \forall v_1, \dots, v_{d-2} \in \mathcal{C}_K^L$
with equality iff neither $K<F<G<L$ nor $K<G<F<L$ holds.

Goal:

1) Prove $\mathrm{pol}_K^L(t)$ is $\mathcal{C}_K^L$-Lorentzian

2) Determine $\alpha, \beta \in \overline{\mathcal{C}_K^L}$ s.t.
$\mathrm{pol}_K^L(x \cdot \alpha + y \cdot \beta)$ gives coeff. of $\overline{\chi}_M(t)$

(Problem: When $C = \mathbb{R}^n_{>0}$, there is a very useful combinatorial characterization of $\mathbb{R}^n_{>0}$-Lorentzian polynomials, but not in general.

Definition of C-Lorentzian too unwieldy: need some useful theory.)

Recall: Lemma: Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be d-homogeneous with $d \geq 3$, and fix $x \in \mathbb{R}^n_{>0}$. If:

(1) $\partial_i p(x) > 0$ for all i,

(2) the Hessian of $\partial_{x_i} p$ at x has exactly one pos. eigenvalue,

(3) the Hessian of p at x is irreducible w/ non-neg. off-diag. entries,

then the Hessian of p at x has exactly one pos. eigenvalue.

Proof:

(1) "Bochner method" implies Hessian satisfies quadratic matrix inequality $\Rightarrow$ no eigenvalues in $(0, \gamma)$

(2) Eigenvector w/ strictly positive entries w/ eigenvalue γ $\Rightarrow$ Exactly one positive eigenvalue by Perron-Frobenius.

(Want to use this to prove some inductive theorem for C-Lorentzian, but what to do about $x > 0$?)

(Given open convex cone, C, its lineality space is $L_C := \overline{C} \cap \overline{(-C)}$ (i.e. largest linear subspace in $\overline{C}$).)

Definition: An open convex cone C is **effective** if $C = C \cap \mathbb{R}^n_{>0} + L_C$ (i.e., $\forall y \in C, \exists x \in C \cap \mathbb{R}^n_{>0}, l \in L_C$ s.t. $y = x + l$).

Theorem: Let $p \in \mathbb{R}[x_1, \dots, x_n]$ be d-homog. with $d \geq 3$, and let C be an open, convex, and effective cone in $\mathbb{R}^n$. If:

(1) $p(x+w) = p(x)$, $\forall x \in \mathbb{R}^n$, $w \in L_C$,

(2) $D_{v_1} \cdots D_{v_d} p > 0$, $\forall v_1, \dots, v_d \in C$,

(3) the Hessian of $D_{v_1} \cdots D_{v_{d-2}} p$ is irreducible w/ non-negative off-diagonal entries, $\forall v_1, \dots, v_{d-2} \in C$,

(4) $\partial_{x_i} p$ is C-Lorentzian, $\forall i$,

then p is C-Lorentzian.

Proof: Fix $v_1, \dots, v_{d-3} \in C$ and consider the cubic $q := D_{v_1} \cdots D_{v_{d-3}} p$. Since $\partial_{x_i} p$ is C-Lorentzian, then so is

$\partial_{x_i} q$ by definition. Fix $y \in C$. By (3),

$$\nabla^2 q(y) \cong \nabla^2 D_y q$$ is irreducible with non-negative off-diag. entries.

Since

$$q(x) = \partial_{t_1} \cdots \partial_{t_{d-3}} \Big|_{t=0} p\Big(x + \sum_{i=1}^{d-3} t_i v_i\Big),$$

it follows from (1) that

$q(x+w) = q(x)$, $\forall x \in \mathbb{R}^n$ and $w \in L_C$.

Thus we may assume WLOG that $y \in \mathbb{R}^n_{>0}$ (by translating by $w \in L_C$) by effectiveness of C. Therefore by the Bochner Lemma, the Hessian of q at y has exactly one positive eigenvalue. Since

$$\nabla^2 q(y) = \nabla^2 [D_{v_1} \cdots D_{v_{d-3}} p](y) \cong \nabla^2 [D_{v_1} \cdots D_{v_3} D_y p],$$

this implies p is C-Lorentzian. $\square$

Want to apply this to $\mathrm{pol}_K^L(t)$

(1) ✓

(2) ✓

(3) non-neg. off-diag. entries ✓

irreducible? means the graph formed by positive off-diag. entries should be connected

↳ vertices are flats and $F \sim G$ if F, G comparable

↳ follows from "semimodularity":

if $a, b \succ a \wedge b$, then $a \vee b \succ a, b$

(4) follows by induction, since C-Lorentzian is preserved under taking products, and since

$\pi_F^G(\mathcal{C}_K^L) \subseteq \mathcal{C}_F^G$ and $\pi_F^G(\mathcal{M}_K^L) \subseteq \mathcal{M}_F^G$

Base case: Write quadratic $\mathrm{pol}_K^L(t)$ explicitly as rank-one minus PSD.

<u>Therefore</u>: $\mathrm{pol}_K^L(t)$ is $\mathcal{C}$-Lorentzian for all $K<L$ in $\mathcal{L}(M)$.

<u>Next</u>: Find $\alpha_K^L, \beta_K^L \in \overline{\mathcal{C}_K^L}$ s.t.

$$\mathrm{pol}_K^L(\alpha\cdot x+\beta\cdot y) =$$

$$\sum_{F \not\ni i} \binom{d(K,L)}{r(K,F)} |\mu(K,F)|\, s^{r(K,F)} t^{d(F,L)}$$

<u>Define</u>: ("almost" submodular, no zero endpoints)

$$\alpha_K^L = \left(\frac{|S\setminus K|}{|L\setminus K|}\right)_{K\subset S\subset L} \qquad \beta_K^L = \left(\frac{|L\setminus S|}{|L\setminus K|}\right)_{K\subset S\subset L}$$

and, given $i\in L\setminus K$, define

$$\alpha_{K,i}^L = (a_S)_{K\subset S\subset L} \qquad \beta_{K,i}^L = (b_S)_{K\subset S\subset L}$$

where $a_S = \begin{cases} 1, & i\in S \\ 0, & i\notin S \end{cases}$, $b_S = \begin{cases} 1, & i\notin S \\ 0, & i\in S \end{cases}$

<u>Facts</u>:

- $\alpha_K^L - \alpha_{K,i}^L \in \mathcal{M}_K^L$, $\beta_K^L - \beta_{K,i}^L \in \mathcal{M}_K^L$
- $K<F<L \Rightarrow$

$$\pi_K^F(\alpha_K^L) = 0,\ \pi_F^L(\alpha_K^L) = \alpha_F^L,\ \pi_K^F(\beta_K^L) = \beta_K^F,\ \pi_F^L(\beta_K^L) = 0$$

(Nice setup for induction.)

<u>Lemma</u>: $\mathrm{pol}_K^L(\alpha_K^L) = \frac{1}{d(K,L)!}$ for $K<L$

<u>Proof</u>: Fix $i \in L \setminus K$. Then $d(K,L) \cdot \mathrm{pol}_K^L(\alpha_K^L)$

$$= \sum_{K<F<L} a_F \cdot \mathrm{pol}_K^F(\pi_K^F(\alpha_K^L)) \cdot \mathrm{pol}_F^L(\pi_F^L(\alpha_K^L))$$

$$= \sum_{K<F<L} a_F \cdot \mathrm{pol}_K^F(0) \cdot \mathrm{pol}_F^L(\alpha_F^L)$$

$$= \sum_{K \lessdot F < L} a_F \cdot \frac{1}{(d(K,L)-1)!}$$ by induction,

since $\mathrm{pol}_K^F(0) \neq 0$ iff $d(K,F)=0$.

Since all $F \in \mathcal{L}(M)$ s.t. $K \lessdot F < L$ partitions $L \setminus K$, we have

$$d(K,L) \cdot \mathrm{pol}_K^L(\alpha_K^L) = \frac{1}{(d(K,L)-1)!}. \qquad \square$$

(What about $\mathrm{pol}_K^L(\beta_K^L)$?)

<u>Theorem (Weisner)</u>: If $K \lessdot F < L$, then $\mu(K,L) = -\sum_{\substack{K<G\lessdot L \\ F \not< G}} \mu(K,G)$.

(implies μ alternates in sign w.r.t. rank)

Lemma: $\mathrm{pol}_K^L(\beta_K^L) = \dfrac{|\mu(K,L)|}{d(K,L)!}$ for $K<L$.

Proof: Fix $i \in L \setminus K$. Then $d(K,L)\cdot \mathrm{pol}_K^L(\beta_K^L)$

$$= \sum_{K<F<L} b_F \cdot \mathrm{pol}_K^F(\beta_K^F)\cdot \mathrm{pol}_F^L(0)$$

$$= \sum_{K<F\lessdot L} b_F \cdot \frac{|\mu(K,F)|}{(d(K,L)-1)!}\cdot 1 \quad \text{by induction}$$

since $\mathrm{pol}_F^L(0)\neq 0$ iff $d(F,L)=0$.

$$= \frac{1}{(d(K,L)-1)!}\sum_{\substack{K<F\lessdot L\\ i\notin F}} |\mu(K,F)|$$

Apply Weisner's theorem with $K \lessdot \mathrm{span}(\{i\}) < L$ to obtain

$$= \frac{1}{(d(K,L)-1)!}\,|\mu(K,L)|,$$

since μ alternates in sign w.r.t. rank.

$\square$

Theorem: If $i\in L\setminus K$, then $\mathrm{pol}_K^L(y\cdot\alpha_K^L + x\cdot\beta_K^L)$

$$= \frac{1}{d(K,L)!}\cdot\sum_{F\not\ni i}\binom{d(K,L)}{r(K,F)}|\mu(K,F)|\cdot x^{r(K,F)}\cdot y^{d(F,L)}.$$

Proof: First we compute

$\partial_x \mathrm{pol}_K^L(y\cdot\alpha_K^L + x\cdot\beta_K^L) =$

$$= \left(D_{\beta_K^L} \operatorname{pol}_K^L\right)\left(y \cdot \alpha_K^L + x \cdot \beta_K^L\right)$$

$$= \sum_{K<F<L} b_F \cdot \operatorname{pol}_K^F(x \cdot \beta_K^F) \cdot \operatorname{pol}_F^L(y \cdot \alpha_F^L)$$

$$= \sum_{\substack{K<F<L \\ i \notin F}} \frac{|\mu(K,F)|\, x^{d(K,F)}}{d(K,F)!} \cdot \frac{y^{d(F,L)}}{d(F,L)}$$

Note further that

$$\operatorname{pol}_K^L(y \cdot \alpha_K^L) = \frac{y^{d(K,L)}}{d(K,L)!}.$$

Thus, $\operatorname{pol}_K^L(y \cdot \alpha_K^L + x \cdot \beta_K^L)$

$$= \frac{1}{d(K,L)!} \sum_{F \not\ni i} \binom{d(K,L)}{r(K,F)} \cdot |\mu(K,F)| \cdot x^{r(K,F)} y^{d(F,L)}$$

since $\mu(K,K) = 1$. $\square$

Since pol_K^L is $\mathcal{C}_K^L$-Lorentzian and $\alpha_K^L, \beta_K^L \in \overline{\mathcal{C}_K^L}$, the coefficients of $\operatorname{pol}_K^L(y \cdot \alpha_K^L + x \cdot \beta_K^L)$ are ultra log-concave. Thus, the coefficients of

$$\overline{\chi_M}(t) = \sum_{F \not\ni i} \mu(K,F) \cdot t^{rk(L) - rk(F) - 1}$$

form an ultra log-concave sequence. This implies the Heron-Rota-Welsh Conjecture.