Lecture 14: Gurvits' theorem (and related results)

Last time: Properties of
$$\mathrm{Cap}_\alpha(p) = \inf_{x>0} \frac{p(x)}{x^\alpha}.$$

Start of proof of Gurvits' Thm.:

If $p \in \mathbb{R}_{\geq 0}(x_1, \ldots, x_n)$ is d-homog. and Lorentzian, then $\forall \mu \in \mathbb{Z}^n_{\geq 0}$, $|\mu| = d$,
$$\mathrm{Cap}_\mu(p) \geq \langle x^\mu \rangle p(x) \geq \binom{d}{\mu} \frac{\mu^\mu}{d^d} \cdot \mathrm{Cap}_\mu(p).$$

Lemma (BLP '20): Let $q, w \in \mathbb{R}^d_{\geq 0}[t]$ be such that w has all pos. coeff. and $\left(\frac{q_k}{w_k}\right)_{k=0}^d$ forms a log-concave sequence (with no holes). Then for all $0 \leq k \leq d$, we have
$$q_k \geq \frac{w_k}{\mathrm{Cap}_k(w)} \cdot \mathrm{Cap}_k(q).$$

Pf.: $\frac{q_i}{w_i} \leq \left(\frac{q_{k+1}}{w_{k+1}}\right)^i$ when $\frac{q_k}{w_k} = 1$.
(also $q_k = 0$ and $q_{k+1} = 0$ cases)

(Need one more lemma.)

Lemma: If $w(t) = (t+1)^d$, then $\mathrm{Cap}_k(w) = \frac{d^d}{k^k (d-k)^{d-k}}$.

(**Note**: $w(t) = (t+1)^d \Rightarrow$ q_k / w_k log-concave $\Leftrightarrow$ q_k ultra log-concave.)

Proof: $\mathrm{Cap}_k(w) = \inf_{t>0} \frac{(1+t)^d}{t^k}$

$$= \left[\inf_{t>0} \frac{1+t}{t^{k/d}} \right]^d = \left[\inf_{t>0} \left(t^{-k/d} + t^{1-k/d} \right) \right]^d$$

$$0 = \partial_t \left[t^{-k/d} + t^{1-k/d} \right] = -\frac{k}{d} t^{-1-\frac{k}{d}} + \left(1 - \frac{k}{d}\right) t^{-\frac{k}{d}}$$

$$= t^{-1-\frac{k}{d}} \left(-\frac{k}{d} + \left(1 - \frac{k}{d}\right) t \right)$$

$$\Rightarrow t = \frac{k/d}{1-k/d} = \frac{k}{d-k}$$

$$\Rightarrow \mathrm{Cap}_k(w) = \frac{\left(1 + \frac{k}{d-k}\right)^d}{\left(\frac{k}{d-k}\right)^k} = \frac{d^d}{k^k (d-k)^{d-k}}. \quad \square$$

Proof of Gurvits' theorem:

Induction on n. ($n=1$ trivial) Now consider:

$$\inf_{x>0} \frac{p(x)}{x^\mu} = \inf_{x_1>0} \cdots \inf_{x_n>0} \frac{p(x_1, \dots, x_n)}{x_1^{\mu_1} \cdots x_n^{\mu_n}}$$

Fix $x_1, \dots, x_{n-1} > 0$. Then:

$$\inf_{x_n>0} \frac{p(x_1,\dots,x_{n-1},x_n)}{x_n^{\mu_n}}$$

$$= \inf_{t>0} \frac{p(x_1,\dots,x_{n-1},t)}{t^{\mu_n}} =: \mathrm{Cap}_{\mu_n}(q)$$

where $q(t) := p(x_1,\dots,x_{n-1},t)$

Defining $f(t,s) := p(x_1 s,\dots,x_{n-1}s,t)$

$f(s,t)$ is Lorentzian degree d,

so the coeff. of q are ultra

log-concave (w.r.t. degree d).

Thus by the lemmas using

$w(t)=(t+1)^d$, we have

$$q_{\mu_n} \geq \frac{w_{\mu_n}}{\mathrm{Cap}_{\mu_n}(w)} \cdot \mathrm{Cap}_{\mu_n}(q)$$

$$= \frac{d!}{\mu_n!(d-\mu_n)!} \cdot \frac{\mu_n^{\mu_n}(d-\mu_n)^{d-\mu_n}}{d^d}$$

$$\cdot \inf_{x_n>0} \frac{p(x_1,\dots,x_{n-1},x_n)}{x_n^{\mu_n}}.$$

Divide by $x_i^{\mu_i}$ $(1 \le i \le n-1)$ and take infs to get:

$$\inf_{x_1,\dots,x_{n-1}>0} \frac{q_{\mu_n}(x_1,\dots,x_{n-1})}{x_1^{\mu_1}\cdots x_n^{\mu_n}} \ge \underbrace{\frac{d!}{\mu_n!(d-\mu_n)!}\cdot\frac{\mu_n^{\mu_n}(d-\mu_n)^{d-\mu_n}}{d^d}}_{C_{\mu_n}} \mathrm{Cap}_\mu(p).$$

Now, $q_{\mu_n}(x_1,\dots,x_{n-1}) = \frac{1}{\mu_n!}\partial_{x_n}^{\mu_n}\big|_{x_n=0} p(x)$

$$\Rightarrow \mathrm{Cap}_{(\mu_1,\dots,\mu_{n-1})}\left(\frac{1}{\mu_n!}\partial_{x_n}^{\mu_n}\big|_{x_n=0} p(x)\right) \ge C_{\mu_n}\cdot \mathrm{Cap}_\mu(p)$$

(Idea of "Capacity preserving operators": derivative operator can only decrease capacity by so much)

Now, by induction,

$$p_\mu = \left\langle x_1^{\mu_1} \cdots x_{n-1}^{\mu_{n-1}} \right\rangle \frac{1}{\mu_n!} \partial_{x_n}^{\mu_n}\Big|_{x_n=0} p(x)$$

$$\geq \binom{d-\mu_n}{\mu_1, \ldots, \mu_{n-1}} \cdot \frac{\mu_1^{\mu_1} \cdots \mu_{n-1}^{\mu_{n-1}}}{(d-\mu_n)^{d-\mu_n}}$$

$$\cdot \operatorname{Cap}_{(\mu_1,\ldots,\mu_{n-1})}\left(\frac{1}{\mu_n!} \partial_{x_n}^{\mu_n}\Big|_{x_n=0} p \right),$$

since $\deg\left(\frac{1}{\mu_n!} \partial_{x_n}^{\mu_n}\Big|_{x_n=0} p \right) = d - \mu_n$.

Thus,

$$p_\mu \geq \frac{(d-\mu_n)!}{\mu_1! \cdots \mu_{n-1}!} \cdot \frac{\mu_1^{\mu_1} \cdots \mu_{n-1}^{\mu_{n-1}}}{(d-\mu_n)^{d-\mu_n}}$$

$$\cdot \frac{d!}{\mu_n! (d-\mu_n)!} \cdot \frac{\mu_n^{\mu_n} (d-\mu_n)^{d-\mu_n}}{d^d} \cdot \operatorname{Cap}_\mu(p).$$

(Simplifying gives the result.) $\square$

(Note that if p real stable, $q(t)$ has ultra log-concave coeff. w.r.t. $\deg(q(t)) \Rightarrow$ better bounds via per-variable deg.)

Corollary: Given $A \in \mathbb{R}_{\geq 0}^{n \times n}$, $\mathrm{Cap}_{\bar{1}}(p_A)$ is an e^n-approx. to $\mathrm{per}(A)$.

Pf: $\mathrm{Cap}_{\bar{1}}(p_A) \geq \mathrm{per}(A) \geq \frac{n!}{n^n} \cdot \mathrm{Cap}_{\bar{1}}(p_A)$

and $\frac{n!}{n^n} \geq e^{-n}$. $\square$

Corollary: Given Lorentzian $p(x)$ in n variables of deg. d., and $\alpha \in \mathbb{Z}_{\geq 0}^n$ s.t. $|\alpha| = d$, and $\alpha_i = \alpha(1)\ \forall i \leq n-1$ then $\mathrm{Cap}_\alpha(p)$ is a simply exponential approx. to $\langle x^\alpha \rangle p(x)$.

Pf.: Stirling's approx.: $\frac{n!}{n^n} = c_n \left(\frac{\sqrt{n}}{e^n} \right)$ for $c_n \in (\sqrt{2\pi}, e]$

$$\Rightarrow \binom{d}{\alpha} \cdot \frac{\alpha^\alpha}{d^d} \geq e^{-n+1} \frac{\sqrt{d}}{e^d} \cdot \prod_{i=1}^n \frac{e^{\alpha_i}}{\sqrt{\alpha_i}}$$

$$= e^{-n+1} \sqrt{d} \cdot \prod_{i=1}^n \frac{1}{\sqrt{\alpha_i}} \geq C^{-n+1}. \quad \square$$

((Can get other bounds for other values of α.)

Mixed Discriminants and Mixed Volumes:

Given $A_1, \dots, A_n \in \mathbb{C}^{n\times n}$, the mixed discriminant of $A_1, \dots, A_n$ is

$$D(A_1, \dots, A_n) := \frac{1}{n!} \partial_{x_1} \cdots \partial_{x_n} \det\left(\sum_k x_k A_k\right).$$

That is $= \langle x^{\mathbb{1}} \rangle \cdot \frac{1}{n!} \det\left(\sum_k x_k A_k\right)$.

For $x_k = a_k + i \cdot b_k$, $b_k > 0$, $a_k \in \mathbb{R}$, and A_k PSD, we have:

$$\det\left(\sum_k x_k A_k\right) = \det(Q + iP)$$

for P pos. def. and Q Hermitian.

Thus $\det(Q+iP)$

$$= \det(P) \cdot \det(i + P^{-1/2} Q P^{-1/2}) \neq 0$$

$\Rightarrow \det\left(\sum_k x_k A_k\right)$ real stable

$\Rightarrow$ Lorentzian.

Thus we have the following:

<u>Cor.</u>: Given PSD $A_1, \dots, A_n$, we have

$$D(A_1, \dots, A_n) \geq \frac{1}{n^n} \operatorname{Cap}_{\mathbb{1}}\left[\det\left(\sum_k x_k A_k\right)\right].$$

hard? easy to evaluate

If $A_i = \operatorname{diag}(a_i)$, then

$$D(A_1, \dots, A_n) = \operatorname{per}\left(\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}\right),$$

since $\det(\sum_k x_k A_k)$ is the corresponding product of linear forms in this case.

<u>D.S. tuple</u>: $\operatorname{tr}(A_k) = 1$, $\sum_k A_k = I$.

<u>Similar</u>: $A_1, \dots, A_n \in \mathbb{C}^{d \times d}$, PSD, $d \geq n \Rightarrow$

$$D(\underbrace{A_1, \dots, A_1}_{\alpha_1}, \underbrace{A_2, \dots, A_2}_{\alpha_2}, \dots, \underbrace{A_n, \dots, A_n}_{\alpha_n}) = \frac{1}{d!} \cdot \partial_x^\alpha \det\left(\sum_k x_k A_k\right).$$

Thus, $D(A_1, \dots, A_1, \dots, A_n, \dots, A_n)$

$$\geq \frac{\alpha^\alpha}{d^\alpha} \cdot \mathrm{Cap}_\alpha \left[\det\left(\sum_k x_k A_k \right) \right]$$

by Gurvits' Theorem again.

Lecture 15 – Denormalized Lorentzian polynomials and Capacity

Last time: Proof of Gurvits' Thm

p Lorentzian d-homog $\Rightarrow$

$\mathrm{Cap}_\alpha(p) \geq \langle x^\alpha \rangle p(x) \geq \binom{d}{\alpha} \frac{\alpha^\alpha}{d^d} \mathrm{Cap}_\alpha(p)$.

Corollary: Bounds on mixed discriminant of PSD matrices:

$$D(\underbrace{A_1, \ldots, A_1}_{\alpha_1}, \underbrace{A_2, \ldots, A_2}_{\alpha_2}, \ldots, \underbrace{A_n, \ldots, A_n}_{\alpha_n})$$

$$= \frac{1}{d!} \cdot \partial_x^\alpha \det\Big(\sum_{k=1}^n x_k A_k\Big)$$

(mistake!) where $A_1, \ldots, A_k$ are $d \times d$ Hermitian PSD matrices.

(generalization of permanent)

Cor (Gurvits): If $\mathrm{tr}(A_k) = 1$ and $\sum_k A_k = \mathrm{Id}$, then

$D(A_1, A_2, \ldots, A_n) \geq \frac{1}{n^n}$.

(Similarly,)

Given convex sets $K_1, \dots, K_n \subseteq \mathbb{R}^n$, the <u>mixed volume</u> of $K_1, \dots, K_n$ is:

$$V(K_1, \dots, K_n) = \frac{1}{n!} \partial_{x_1} \cdots \partial_{x_n} \mathrm{Vol}\Big(\sum_i x_i K_i\Big).$$

<u>Minkowski</u>: $\mathrm{Vol}(\sum_i x_i K_i)$ is a n-homogeneous polynomial in $x_1, \dots, x_n$.

$\hookrightarrow$ Lorentzian by HW.

<u>Cor.</u>: Given convex sets $K_1, \dots, K_n \subseteq \mathbb{R}^n$, we have

$$\underset{\uparrow\ \text{harder}}{V(K_1, \dots, K_n)} \geq \frac{1}{n^n} \mathrm{Cap}_{\mathbb{1}}\Big[\mathrm{Vol}\Big(\underset{\uparrow\ \text{hard}}{\sum_i x_i K_i}\Big)\Big]$$

<u>Similarly</u>: $K_1, \dots, K_n \in \mathbb{R}^d$, $n \leq d \Rightarrow$

$$V(\underbrace{K_1, \dots, K_1}_{\alpha_1}, \dots, \underbrace{K_n, \dots, K_n}_{\alpha_n}) \geq \frac{\alpha^\alpha}{d^d} \mathrm{Cap}_\alpha\Big[\mathrm{Vol}\Big(\sum_i x_i K_i\Big)\Big].$$

(Gives simply exponential approx. to mixed volumes, given an oracle for the volume of Minkowski sums.)

<u>Theorem (BKK '75)</u>: Given Newton polytopes $P_1, \dots, P_n$, then the number of solutions to a generic polynomial system $f_1 = f_2 = \dots = f_n = 0$ where $\mathrm{Newt}(f_K) = P_K$ is equal to $n! \cdot V(P_1, \dots, P_n)$.

(<u>Cor.</u>: If $(P_1, \dots, P_n)$ is "d-regular", then the number of solutions is bounded below by $d^n \cdot \frac{n!}{n^n}$.)

(<u>Q</u>: What does "d-regular" mean here? Some type of scaled doubly stochastic.)

Denormalized Lorentzian polys.

Def.: A d-homog. polynomial $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is called <u>denormalized Lorentzian</u> (DL) if

$$N[p] := \sum_{\kappa \in \mathbb{Z}_{\geq 0}^n} p_\kappa \frac{x^\kappa}{\kappa!} \cong \sum_{\kappa \in \mathbb{Z}_{\geq 0}^n} \binom{d}{\kappa} p_\kappa x^\kappa.$$

is Lorentzian.

Examples: Schur polynomials, conj. Schubert polynomials, [HMMS] contingency table generating polys., [BLP] more?

Properties: If p, q are DL, then so are the following:

(1) $\partial_{x_1}^{\kappa}|_{x_1=0}\, p$

(2) $p \cdot q$

(3) $p(\alpha x_1, \beta x_1, x_3, \dots, x_n)$ $\forall \alpha, \beta > 0$.

Proof:

(1) $p(x) = \sum_{i=0}^{d} x_1^i \, p_i(x_2, \dots, x_n)$

$\Rightarrow \partial_{x_1}^k |_{x_1=0} \, p(x) = k! \cdot p_k(x_2, \dots, x_n)$

$\Rightarrow \partial_{x_1}^k |_{x_1=0} \, N[p](x)$

$= N[\partial_1^k |_{x_1=0} \, p](x)$

up to pos. scalar.

(2) Note that products in disjoint variables preserves DL, since N is a per-variable operation. (up to scalar)

Thus, (3) implies (2) via

$p(x_1, \dots, x_n) \, q(z_1, \dots, z_n)$

$\mapsto p(x_1, \dots, x_n) \cdot q(x_1, \dots, x_n).$

(3) Scaling commutes w/ N, so we just need to show

$T: p \mapsto p(x_1, x_1, x_3, \ldots, x_n)$

preserves DL, which is iff

$N \circ T \circ N^{-1}$ preserves Lorentzian.

(HW problem) $\square$

<u>Goal</u>: Generalize Gurvits' thm. to DL polynomials:

<u>Theorem (BLP '20)</u>: If $p \in \mathbb{R}^{\lambda}_{\geq 0}[x_1, \ldots, x_n]$ is d-homog. DL polyn. in n variables, then $\forall \alpha \in \mathbb{Z}^n_{\geq 0}$, $|\alpha| = d$, we have

$$\mathrm{Cap}_\alpha(p) \geq \langle x^\alpha \rangle p(x) \geq \left[\prod_{i=2}^{n} \frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}}\right] \mathrm{Cap}_\alpha(p).$$

Actually can replace each term by

$$\max\left\{ \frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}}, \frac{(\lambda_i - \alpha_i)^{d-\alpha_i}}{(\lambda_i - \alpha_i + 1)^{d-\alpha_i+1}} \right\}$$

where λ_i is per-variable max deg.

Lemma: If $p(x,y)$ is bivariate and DL, then the coeff. of p form a log-concave sequence without holes.

(**Pf.:** Immediate from Lorentzian bivariate.)

Lemma: If $q(t) \in \mathbb{R}^d_{\geq 0}[t]$ has log-concave coeff. (w/ no holes), then

$$q_k \geq \max\left\{ \frac{k^k}{(k+1)^{k+1}}, \frac{(d-k)^{d-k}}{(d-k+1)^{d-k+1}} \right\} \cdot \mathrm{Cap}_k(q).$$

Proof: By reversing coeff. order, we only need to show

$$q_k \geq \frac{k^k}{(k+1)^{k+1}} \cdot \mathrm{Cap}_k(q).$$

By prev. Lemma, need to show for $w(t) = 1+t+t^2+\dots+t^d$ that

$$\frac{w_k}{\mathrm{Cap}_k(w)} \geq \frac{k^k}{(k+1)^{k+1}}$$

$$\iff \mathrm{Cap}_k(w) \leq \frac{(k+1)^{k+1}}{k^k}$$

We have $\mathrm{Cap}_k(w) = \inf_{t>0} \sum_{i=0}^{d} t^{i-k}$

$$\leq \inf_{t>0} \sum_{i=0}^{\infty} t^{i-k} = \inf_{0<t<1} t^{-k}(1-t)^{-1}$$

$$= \left[\sup_{0<t<1} t^k - t^{k+1}\right]^{-1}$$

$$0 = \partial_t\left[t^k - t^{k+1}\right] = k t^{k-1} - (k+1) t^k$$

$$= t^{k-1}\left[k - (k+1)t\right] \Rightarrow t = \frac{k}{k+1}$$

$$\Rightarrow \mathrm{Cap}_k(w) \leq \left(\frac{k}{k+1}\right)^k \left(1 - \frac{k}{k+1}\right)^{-1}$$

$$= \frac{(k+1)^{k+1}}{k^k}. \quad \square$$

(<u>Proof of Coeff. bound for DL</u>:

Same as Gurvits' theorem, since $\partial_{x_i}^k|_{x_i=0}$ and $p \mapsto p(x_1 s, \ldots, x_{n-1} s, t)$, $x_i > 0$ preserve DL, and log-concavity doesn't depend on degree, unlike ultra log-concavity.)

Lecture 16: DL polynomials and Transportation polytopes

Last time: DL polynomials:

p DL $\Longleftrightarrow$ $N[p]$ Lorentzian,

$$N[x^k] = N[x_1^{k_1} \cdots x_n^{k_n}] = \frac{x_1^{k_1} \cdots x_n^{k_n}}{k_1! \cdots k_n!} = \frac{x^k}{k!}$$

Properties: p, q DL $\Rightarrow$ So are:

(1) $\partial_{x_1}^{\alpha_1}|_{x_1=0}\, p$

(2) $p \cdot q$

(3) $p(ax_1, bx_1, x_3, \ldots, x_n)$, $a, b \geq 0$.

Lemma: $q \in \mathbb{R}^d_{\geq 0}[t]$ has log-concave coefficients (w/ no holes) $\Longrightarrow$

$$q_k \geq \max\left\{ \frac{k^k}{(k+1)^{k+1}}, \frac{(d-k)^{d-k}}{(d-k+1)^{d-k+1}} \right\} \cdot \mathrm{Cap}_k(q).$$

Thm.: For DL $p \in \mathbb{R}^{\lambda}_{\geq 0}[x_1, \ldots, x_n]$ and $\alpha \in \mathbb{Z}^n_{\geq 0}$,

$$\mathrm{Cap}_\alpha(p) \geq \langle x^\alpha \rangle p(x) \geq \left[\prod_{i=2}^{n} C^{\lambda_i}_{\alpha_i} \right] \mathrm{Cap}_\alpha(p),$$

where $C^d_k = \max\left\{ \frac{k^k}{(k+1)^{k+1}}, \frac{(d-k)^{d-k}}{(d-k+1)^{d-k+1}} \right\}$.

Today – proof: Same as for Gurvits' thm..

$$\frac{1}{\alpha_1!}\partial_{x_1}^{\alpha_1}\Big|_{x_1=0}\cdots\frac{1}{\alpha_n!}\partial_{x_n}^{\alpha_n}\Big|_{x_n=0} p = p_\alpha$$

If $n=1$, trivial by homog. (w/ const. 1)

Define $f(x_1,\dots,x_{n-1}) := \frac{1}{\alpha_n!}\partial_{x_n}^{\alpha_n}\Big|_{x_n=0} p$

By induction on n,

$$p_\alpha = f_{(\alpha_1,\dots,\alpha_n)} \geq \left[\prod_{i=2}^{n-1} C_{\alpha_i}^{\lambda_i}\right]\cdot \mathrm{Cap}_{(\alpha_1,\dots,\alpha_{n-1})}(f)$$

Now fix $x_1,\dots,x_{n-1}>0$ and consider $p(x_1 s, x_2 s,\dots,x_{n-1}s, t)$, which is DL $\Rightarrow$ coeff are log-concave.

Thus $q(t) := p(x_1,x_2,\dots,x_{n-1},t) \in \mathbb{R}^{\lambda_n}_{\geq 0}[t]$ has log-concave coeff. also.

(Key difference from Lorentzian, but also works for real stable)

Thus, $q_{\alpha_n} \geq C_{\alpha_n}^{\lambda_n}\cdot \mathrm{Cap}_{\alpha_n}(q)$.

Now,

$$q_{\alpha_n} = \frac{1}{\alpha_n!} \partial_t^{\alpha_n}\Big|_{t=0} q(t)$$
$$= f(x_1, \dots, x_{n-1})$$

$$\Rightarrow f(x_1, \dots, x_{n-1}) \geq C_{\alpha_n}^{\lambda_n} \cdot \inf_{x_n > 0} \frac{p(x_1, \dots, x_n)}{x_n^{\alpha_n}}$$

$\forall x_1, \dots, x_{n-1} > 0$.

Dividing by $x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}$ and taking infs implies

$$\mathrm{Cap}_{(\alpha_1, \dots, \alpha_{n-1})}(f) \geq C_{\alpha_n}^{\lambda_n} \cdot \mathrm{Cap}_\alpha(p).$$

Combining with the previous observation gives

$$p_\alpha \geq \left[\prod_{i=2}^{n} C_{\alpha_i}^{\lambda_i}\right] \cdot \mathrm{Cap}_\alpha(p). \qquad \square$$

(Note how $n=2$ case corresponds to homog. bivariate, and we obtain the result from the q/w lemma, as expected.)

Contingency tables and transportation polytopes

Def.: Fix $m, n \in \mathbb{Z}_{>0}$, $\alpha \in \mathbb{Z}^m_{\geq 0}$, $\beta \in \mathbb{Z}^n_{\geq 0}$, and let $T(\alpha, \beta)$ denote the polytope of all matrices $M \in \mathbb{R}^{m \times n}_{\geq 0}$ with row sums α and column sums β. $T(\alpha, \beta)$ is called a <u>transportation polytope</u>.

<u>Goal</u>: Approximate/bound the number of integer points of $T(\alpha, \beta)$, called <u>contingency tables</u> (maybe volume too).

<u>Generating function</u>:

$$G(x,y) = \prod_{i=1}^{m} \prod_{j=1}^{n} \sum_{k=0}^{\infty} x_i^k y_j^k$$

Why? Each term in expanded sum corresponds to a non-neg. integer matrix, and the exponent vectors give row sums (x) and column sums (y).

Thus $G(x,y) = \sum_{\alpha,\beta} \#CT(\alpha,\beta) \cdot x^\alpha y^\beta$

$$= \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1-x_i y_j}.$$

(Problems: Not a polynomial, and not homogeneous.)

Fix α, β s.t. $|\alpha|, |\beta| \leq d$, and define:

$$G_d(x,y) = \prod_{i=1}^{m} \prod_{j=1}^{n} \sum_{k=0}^{d} x_i^k y_j^k$$

$$\tilde{G}_d(x,y) = \prod_{i=1}^{m} \prod_{j=1}^{n} \sum_{k=0}^{d} x_i^k y_j^{d-k}$$

s.t. $G_d(x,y) = y^{d \cdot \mathbb{1}} \cdot \tilde{G}_d(x, y^{-1})$ is a truncation of $G(x,y)$.

Lemma: $\tilde{G}_d(x,y)$ is DL, $\forall d$.

Proof: Log-concave coeff plus products

Thus, by the theorem,

$$\langle x^{\alpha} y^{d\cdot\bar{1}-\beta}\rangle \tilde{G}_d(x,y)$$

$$\geq \prod_{i=2}^{m} \frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}} \cdot \prod_{j=1}^{n} \frac{\beta_j^{\beta_j}}{(\beta_j+1)^{\beta_j+1}} \cdot \operatorname{Cap}_{(\alpha, d\cdot\bar{1}-\beta)}(\tilde{G}_d)$$

$$= \inf_{x,y>0} \frac{\tilde{G}_d(x,y)}{x^{\alpha} y^{d\cdot\bar{1}-\beta}} = \inf_{x,y>0} \frac{\tilde{G}_d(x,y^{-1})}{x^{\alpha} y^{\beta-d\cdot\bar{1}}}$$

$$= \inf_{x,y>0} \frac{y^{d\cdot\bar{1}}\cdot \tilde{G}_d(x,y^{-1})}{x^{\alpha} y^{\beta}}$$

Further, $(= G_d(x,y))$

$$\langle x^{\alpha} y^{d\cdot\bar{1}-\beta}\rangle \tilde{G}_d(x,y)$$

$$= \langle x^{\alpha} y^{\beta}\rangle G(x,y)$$

$\Rightarrow$ Bound holds for all d.

That is,

$$\langle x^{\alpha} y^{\beta}\rangle G(x,y)$$

$$\geq \prod_{i=2}^{m} \frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}} \prod_{j=1}^{n} \frac{\beta_j^{\beta_j}}{(\beta_j+1)^{\beta_j+1}} \cdot \operatorname{Cap}_{\alpha,\beta}(G_d)$$

for all $d \geq |\alpha|, |\beta|$.

<u>Want:</u> $\langle x^\alpha y^\beta \rangle G(x,y)$

$\geq C_{\alpha,\beta} \cdot Cap_{\alpha,\beta}(G)$ where

$\left(G(x,y) = \text{"}\lim_{d\to\infty}\text{"}\, y^{d\cdot \bar{1}}\, \tilde{G}_d(x, y^{-1}) \right).$

<u>Need to show:</u>

$$\lim_{d\to\infty} Cap_{(\alpha,\beta)}(G_d) = Cap_{(\alpha,\beta)}(G).$$

$$\Leftrightarrow \lim_{d\to\infty}\left[\inf_{x,y>0} \frac{\prod_{i,j}\sum_{k=0}^{d} x_i^k y_j^k}{x^\alpha y^\beta}\right] = \inf_{x,y>0} \frac{\prod_{i,j}\sum_{k=0}^{\infty} x_i^k y_j^k}{x^\alpha y^\beta}$$

(actually, $1 > x_i y_j > 0$.)

<u>Proof:</u> $\lim_{d\to\infty} \dfrac{\prod_{i,j}\sum_{k=0}^{d} x_i^k y_j^k}{x^\alpha y^\beta} = \dfrac{\prod_{i,j}\sum_{k=0}^{\infty} x_i^k y_j^k}{x^\alpha y^\beta}$

is uniform on $\{(x,y) : \varepsilon < x_i y_j < 1-\varepsilon \ \forall i,j\}$

for all $\varepsilon > 0$, and for $\alpha, \beta > 0$ and large enough d,

$\dfrac{\prod_{i,j}\sum_{k=0}^{d} x_i^k y_j^k}{x^\alpha y^\beta}$ cannot be

infimized when $x_i y_j$ near boundary.

Thus we can restrict to the domain $\{(x,y) : \varepsilon < x_i y_j < 1-\varepsilon \ \forall i,j\}$ for some $\varepsilon > 0$, and lim and inf can be exchanged when convergence is uniform. $\square$

<u>Theorem</u>: For all m, n, $\alpha, \beta > 0$, we have that

$$\mathrm{Cap}_{\alpha,\beta}(G) \geq \#CT(\alpha,\beta)$$

$$\geq \prod_{i=2}^{m} \frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}} \prod_{j=1}^{n} \frac{\beta_j^{\beta_j}}{(\beta_j+1)^{\beta_j+1}} \cdot \mathrm{Cap}_{\alpha,\beta}(G)$$

$$\geq e^{-m-n+1} \prod_{i=2}^{m} \frac{1}{\alpha_i+1} \prod_{j=1}^{n} \frac{1}{\beta_j+1} \cdot \mathrm{Cap}_{\alpha,\beta}(G).$$

(Since $\left(\frac{t}{t+1}\right)^t \geq \frac{1}{e}$ for $t > 0$)

Lecture 17:

Last time: Bounds on # of contingency tables in $T(\alpha,\beta) := \{M \in \mathbb{R}_{\geq 0}^{m\times n} : M\cdot\mathbb{1} = \alpha,\ M^T\cdot\mathbb{1} = \beta\}$.

Theorem: $\langle x^\alpha y^\beta \rangle G(x,y)$

$$\geq \prod_{i=2}^{m} \frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}} \prod_{j=1}^{n} \frac{\beta_j^{\beta_j}}{(\beta_j+1)^{\beta_j+1}} \cdot \mathrm{Cap}_{\alpha,\beta}(G)$$

$$\geq e^{-m-n+1} \prod_{i=2}^{m} \frac{1}{\alpha_i+1} \prod_{j=1}^{n} \frac{1}{\beta_j+1} \cdot \mathrm{Cap}_{\alpha,\beta}(G).$$

(Since $\left(\frac{t}{t+1}\right)^t \geq \frac{1}{e}$ for $t>0$)

where $G(x,y) = \prod_{i=1}^{m}\prod_{j=1}^{n} \frac{1}{1-x_iy_j}$

$$= \sum_{\alpha,\beta\geq 0} \#CT(\alpha,\beta)\, x^\alpha y^\beta.$$

(**Two more ideas**: limit to get volume, and explicit capacity expression for specific values of α,β.)

E.g.: Uniform transportation polytopes

$\wp$ $\alpha = \alpha_0 \cdot \mathbb{1}$ and $\beta = \beta_0 \cdot \mathbb{1}$. Then

$\alpha_0 \cdot m = \beta_0 \cdot n$, or else $T(\alpha, \beta) = \emptyset$.

(Q: Can we compute capacity explicitly in this case?)

E.g.: $\alpha_0 = \beta_0 = 1 \Rightarrow T(\alpha, \beta)$ is the Birkhoff polytope.

(Q: Can we get a bound on the volume in this case?)

Asymptotics are known, so we can compare.

(First we need a polarization-type result for capacity. This foreshadows what we will do next lecture in combining capacity and stability/Lorentzian preservers.)

Lemma: $S\ p(x_1,\dots,x_m,y_1,\dots,y_n)$ symmetric in y_i and $\alpha\in\mathbb{R}^m_{\geq 0}$ and $\beta\in\mathbb{R}^n_{\geq 0}$ with $\beta=\beta_0\cdot\mathbb{1}$. Then

$$\mathrm{Cap}_{\alpha,\beta}(p)=\mathrm{Cap}_{(\alpha,\beta_0\cdot n)}\big(p(x_1,\dots,x_m,y,\dots,y)\big).$$

Proof: Fix $x_1,\dots,x_m>0$, define

$$f(y_1,\dots,y_n):=p(x_1,\dots,x_m,y_1,\dots,y_n).$$

Consider $\inf\limits_{y_j>0}\dfrac{f(y_1,\dots,y_n)}{y_1^{\beta_0}\cdots y_n^{\beta_0}}=$

$$\underset{(y_j=e^{z_j})}{\exp}\left\{\inf_{z_j\in\mathbb{R}}\left[\log f(e^{z_1},\dots,e^{z_n})-\beta_0\cdot\langle z,\mathbb{1}\rangle\right]\right\}$$

Objective function is convex and symmetric in z_j.
This implies it is infimized on $z_1=z_2=\dots=z_n$. Thus

$$\inf_{y_j>0}\frac{f(y_1,\dots,y_n)}{y_1^{\beta_0}\cdots y_n^{\beta_0}}=\inf_{y>0}\frac{f(y,\dots,y)}{y^{n\cdot\beta_0}}\Rightarrow$$

$$\inf_{x_i,y_j>0}\frac{p(x_1,\dots,x_m,y_1,\dots,y_n)}{x_1^{\alpha_1}\cdots x_m^{\alpha_m}y_1^{\beta_0}\cdots y_n^{\beta_0}}=\inf_{x_i,y>0}\frac{p(x_1,\dots,x_m,y,\dots,y)}{x_1^{\alpha_1}\cdots x_m^{\alpha_m}y^{n\cdot\beta_0}}.$$

by dividing by x^α and taking infs. $\square$

Now, $G(x,y) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1-x_i y_j}$

is symmetric in x_i and y_j separately.

(Note also that $\log G(e^x, e^y)$ is convex as required for the proof of the previous lemma.)

Thus, $\mathrm{Cap}_{\alpha_0 \cdot \vec{1}, \beta_0 \cdot \vec{1}}(G)$

$$= \inf_{t,s>0} \frac{(1-ts)^{-mn}}{t^{m \cdot \alpha_0} s^{n \cdot \beta_0}}$$

Recall $m \cdot \alpha_0 = n \cdot \beta_0 = C$

$\Rightarrow$ above is symmetric in t, s

$$= \inf_{t>0} \frac{(1-t^2)^{-mn}}{t^{2C}} \qquad (t^2 \to t)$$

$$= \inf_{t>0} \frac{(1-t)^{-mn}}{t^C} = \left[\sup_{t>0} t^{\frac{C}{mn}}(1-t)\right]^{-mn}$$

$$= \underset{\substack{\uparrow \\ \text{calculus} \\ \text{from before}}}{\cdots} = \frac{(C+mn)^{C+mn}}{(mn)^{mn} \cdot C^C}.$$

Therefore

$$\#CT(\alpha,\beta) \geq \left(e(\alpha_0+1)\right)^{-m+1}\left(e(\beta_0+1)\right)^{-n} \cdot \frac{(n\cdot\beta_0+mn)^{n\cdot\beta_0+mn}}{(mn)^{mn}(n\cdot\beta_0)^{n\beta_0}}$$

$$= e^{-m-n+1}(\alpha_0+1)^{-m+1}(\beta_0+1)^{-n} \cdot \left[\frac{(\beta_0+m)^{\beta_0+m}}{m^m\,\beta_0^{\beta_0}}\right]^n .$$

Q: What about volume?

E.g. Birkhoff polytope, $\alpha=\beta=\bar{1}$ $(m=n)$

Key idea:

$$\mathrm{vol}(T(\bar{1},\bar{1})) \cong \lim_{d\to\infty} \frac{\#CT(d\cdot\bar{1},d\cdot\bar{1})}{d^{(n-1)^2}}$$

(Why? Dimension is $(n-1)^2$, so this picks out the leading coeff of Ehrhart polynomial, which is volume up to scalar.)

$$\underline{\text{Now}}: \quad \lim_{c\to\infty} \frac{\#CT(c\cdot I, c\cdot I)}{c^{(n-1)^2}}$$

$$\geq \lim_{c\to\infty} \frac{e^{1-2n}}{c^{n^2-2n+1}} \prod_{i=2}^{n} \frac{1}{c+1} \prod_{j=1}^{n} \frac{1}{c+1} \cdot \mathrm{Cap}_{(c\cdot I, c\cdot I)}(G)$$

$$= e^{1-2n} \cdot \inf_{x,y>0} \left[\frac{\prod_{i,j=1}^{n} \frac{1}{c} \sum_{k=0} (x_i y_j)^k}{x_1^c \cdots x_n^c \, y_1^c \cdots y_n^c} \right]$$

$$= e^{1-2n} \inf_{x,y>0} \left[\frac{\prod_{i,j=1}^{n} \frac{1}{c} \sum_{k=0}^{c} (x_i y_j)^{k/c}}{x_1 \cdots x_n \, y_1 \cdots y_n} \right] \qquad \begin{matrix} x_i \mapsto x_i^{1/c} \\ y_j \mapsto y_j^{1/c} \end{matrix}$$

$$\hookrightarrow \approx \int_0^\infty (x_i y_j)^t \, dt$$

$$= \left[\frac{(x_i y_j)^t}{\ln(x_i y_j)} \right]_0^\infty = \frac{-1}{\ln(x_i y_j)} \quad \text{for } x_i y_j < 1.$$

$$\Rightarrow \mathrm{Vol}(T(I,I)) \geq e^{1-2n} \cdot \mathrm{Cap}_{(I,I)}\left(\prod_{i,j=1} \frac{-1}{\ln(x_i y_j)} \right)$$

$$\hookrightarrow = \mathrm{Cap}_{(n,n)} \left[\left(\frac{-1}{\ln(xy)} \right)^{n^2} \right]$$

$$= \mathrm{Cap}_n \left[\left(\frac{-1}{\ln x} \right)^{n^2} \right] = \left[\mathrm{Cap}_{1/n} \left(\frac{-1}{\ln x} \right) \right]^{n^2}$$

$$\mathrm{Cap}_{1/n}\left(\frac{-1}{\ln x} \right) = \left[\sup_{1>x>0} (-\ln x) x^{1/n} \right]^{-1}$$

$$0 = \partial_x = \frac{-1}{x} \cdot x^{1/n} - \frac{1}{n} \ln x \cdot x^{1/n - 1}$$

$$= -x^{1/n-1} \left[1 + \frac{1}{n} \ln x \right] \Rightarrow x = e^{-n}.$$

$$\Rightarrow \mathrm{Vol}(T(I,I)) \geq e^{1-2n} \cdot \left(\frac{e}{n} \right)^{n^2} = e^{(n-1)^2} n^{-n^2}$$

$\underline{\text{Exact asymp}}$: $\sim C \cdot (2\pi)^{-n} e^{n^2 + O(1)} n^{-n^2+n}$ [CM'09]

(Key: Factor of $(c+1)$ in our bound (power $= 2n-1$) is precisely what was needed.)

(Last Comments about Coeff. bounds via capacity.)

Major recent application: Metric TSP

How? Suppose $p(\bar{1}) = 1$. Then $p_\alpha = \mathbb{P}[\mu = \alpha]$ for some discrete distribution μ. Weighted distributions on spanning trees have real stable gen. poly. $\Rightarrow$ lower bounds on probabilities. Since distribution is finite/discrete, this implies some property of the expected random tree.

E.g.: n-homog. n-variate real stable $\Rightarrow$ $p_{\bar{1}} \geq \frac{n!}{n^n} \mathrm{Cap}_{\bar{1}}(p)$.

Thm. (GL '21): If $\|\bar{1} - \nabla p(\bar{1})\|_1 < 2$ and •, then

$$\mathrm{Cap}_{\bar{1}}(p) \geq \left(1 - \frac{\|\bar{1} - \nabla p(\bar{1})\|_1}{2}\right)^n.$$

Lecture 18:

Last time: Finished coeff bounds.

- integer points of transportation polytopes.

$$\#CT(\alpha,\beta) \geq e^{1-m-n} \prod_{i=2}^{m} \frac{1}{\alpha_i+1} \prod_{j=1}^{n} \frac{1}{\beta_j+1} \cdot \mathrm{Cap}_{\alpha,\beta}(G)$$

- explicit bound for symmetric case
  (via symmetric capacity lemma)
- Volume bounds via limiting.
- Also: explicit bounds for $\mathrm{Cap}_\alpha(p)$ can be achieved when $\mathbb{E}[\mu] \approx \alpha$.
  (μ is distr. given by p)

(Next: Capacity bounds on more complicated quantities related to real stable and Lorentzian polynomials)

Bilinear Forms and Capacity preserving Operators

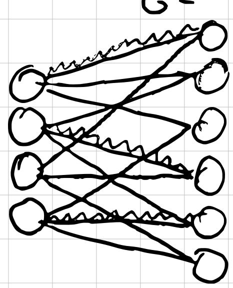
Motivation: Let G be an (a,b)-biregular (m,n)-bipartite graph (with $am=bn$), and consider the $m\times n$ bipartite adjacency matrix $A \in \{0,1\}^{m\times n}$. Define

$$p_G(x) = \prod_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j .$$

Goal: Count size-k matchings.

E.g.:

$G =$



$A =$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

"(3,2)-stochastic"

$p_G =$

$$(x_1+x_2+x_3)\cdot(x_2+x_4+x_5)\cdot(x_1+x_4+x_6)\cdot(x_3+x_5+x_6)$$

\# size-3 matchings $\Leftrightarrow$ size-3 subpermanent of A

$$\mu_k(G) \sim \sum_{S \in \binom{[n]}{k}} \partial_x^S p_G(x)$$

<u>Problem</u>: Remaining polyn. has deg. $m-k$.

By regularity, set $x = \overline{1}$ after and divide by a^{m-k} (a = row sum):

$$\mu_k(G) = \frac{1}{a^{m-k}} \sum_{S \in \binom{[n]}{k}} \partial_x^S p_G(x) \Big|_{x=\overline{1}}.$$

<u>New issue</u>: $\sum_{S \in \binom{[n]}{k}} \partial_x^S p_G(x)$ does not pick out a coefficient.

<u>Another motivation: Matroid intersection</u>

p_{M_1}, p_{M_2} Lorentzian, but

$$p_{M_1 \cap M_2}(x) = \sum_{B \in M_1 \cap M_2} x^B$$

not necessarily Lorentzian

<u>But</u>: Maybe we can count:

$|M_1 \cap M_2| = \langle p_{M_1}, p_{M_2} \rangle_2$, Hadamard inner product on coeff.

Idea: Certain bilinear form w/ stability/Lorentzian preserving info. was used to give stability/Lorentzian preserver theorems. Can we do the "same" for capacity?

Define: For $p, q \in \mathbb{R}^{\lambda}_{\geq 0}[x_1, \ldots, x_n]$,

$$\langle p, q \rangle^{\lambda} := \sum_{0 \leq \kappa \leq \lambda} \prod_{i=1}^{n} \binom{\lambda_i}{\kappa_i}^{-1} \cdot p_\kappa q_{\lambda - \kappa}$$

$$\Rightarrow \langle p, q \rangle^{\mathbb{1}} = \sum_{S \subseteq [n]} p_S q_{S^c}$$

$$= \prod_{i=1}^{n} (\partial_{x_i} + \partial_{z_i})\Big|_{x_i = z_i = 0} p(x) q(z)$$

($\langle p, q \rangle^{\mathbb{1}}$ related to BB char.

↳ capacity bound on $\langle p, q \rangle^{\mathbb{1}}$?

↳ translate to linear operators?)

Lemma (side board): $|\alpha| \leq d$, $\alpha_0 := d - |\alpha| \Rightarrow$

$$\mathrm{Cap}_\alpha((a_0 + a_1 x_1 + \cdots + a_n x_n)^d) = \prod_{i=0}^{n} \left(\frac{d \cdot a_i}{\alpha_i} \right)^{\alpha_i}.$$

Pf: Homogenize, then calculus.

Theorem (AOG '17): $p, q \in \mathbb{R}^{(1,\dots,1)}_{\geq 0}[x_1,\dots,x_n]$ real stable and $\alpha \in [0,1]^n \Rightarrow$

$$\langle p, q \rangle^{\bar{1}} \geq \alpha^{\alpha}(\bar{1}-\alpha)^{\bar{1}-\alpha} \cdot Cap_{\alpha}(p) \cdot Cap_{\bar{1}-\alpha}(q)$$

Proof:

$$\langle p, q \rangle^{\bar{1}} = \prod_{i=1}^{n} (\partial_{x_i} + \partial_{z_i})\Big|_{x_i = z_i = 0} p(x) q(z)$$

$$Cap_{\alpha}(p) \cdot Cap_{\bar{1}-\alpha}(q) = Cap_{(\alpha, \bar{1}-\alpha)}(p(x) q(z))$$

Proof by induction on n.

$$\prod_{i=1}^{n-1} (\partial_{x_i} + \partial_{z_i})\Big|_{x_i = z_i = 0} p(x) q(z)$$
$$= a x_n z_n + b x_n + c z_n + d$$

So, $\forall x_n, z_n > 0$, $f(x_1,\dots,x_{n-1}) = p(x_1,\dots,x_n)$
$g(z_1,\dots,z_{n-1}) = q(x_1,\dots,x_n)$

$$\Rightarrow \langle f, g \rangle^{\bar{1}} \geq \alpha_1^{\alpha_1} \cdots \alpha_{n-1}^{\alpha_{n-1}} (1-\alpha_1)^{1-\alpha_1} \cdots (1-\alpha_{n-1})^{1-\alpha_{n-1}}$$
$$\cdot Cap_{(\alpha_1,\dots,\alpha_{n-1}, 1-\alpha_1,\dots,1-\alpha_{n-1})}(f(x) g(z))$$

Divide by $x_n^{\alpha_n} z_n^{1-\alpha_n}$ and take inf $\Rightarrow$

$$Cap_{\alpha_n, 1-\alpha_n}(a x_n z_n + b x_n + c z_n + d)$$
$$\geq \alpha_1^{\alpha_1} \cdots \alpha_{n-1}^{\alpha_{n-1}} (1-\alpha_1)^{1-\alpha_1} \cdots (1-\alpha_{n-1})^{1-\alpha_{n-1}} Cap_{\alpha, \bar{1}-\alpha}(p(x) q(z)).$$

Now, $\langle p, q \rangle^{I}$

$$= (\partial_{x_n} + \partial_{z_n})|_{x_n = z_n = 0} (a x_n z_n + b x_n + c z_n + d)$$

$$= b + c \implies$$

<u>To show</u>: Base case $n = 1$

$$b + c \geq \alpha_n^{\alpha_n} (1-\alpha_n)^{1-\alpha_n} \cdot \mathrm{Cap}_{\alpha_n, 1-\alpha_n}(a x_n z_n + b x_n + c z_n + d)$$

<u>Recall</u>: Real stable $h \in \mathbb{R}^{(1,1)}[x_n, z_n]$

iff $\partial_{x_n} h \cdot \partial_{z_n} h \geq h \cdot \partial_{x_n} \partial_{z_n} h$

iff $b \cdot c \geq a \cdot d$.

(Assume $a > 0$. Other case similar.)

$a x z + b x + c z + d$ (drop indices for simplicity)

$$\leq a x z + b x + c z + \frac{bc}{a} \qquad (a, b, c, d \geq 0)$$

$$= (a z + b)\left(x + \frac{c}{a}\right)$$

$$\implies \mathrm{Cap}_{\alpha, 1-\alpha}(a x z + b x + c z + d)$$

$$\leq \inf_{x > 0} \frac{x + \frac{c}{a}}{x^{\alpha}} \cdot \inf_{z > 0} \frac{a z + b}{z^{1-\alpha}}$$

$$= \frac{1^{\alpha} \left(\frac{c}{a}\right)^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \cdot \frac{a^{1-\alpha} b^{\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \quad \text{(via Lemma.)}$$

Now: $\mathrm{Cap}_{\alpha,1-\alpha}(axz+bx+cz+d)$

$$\le \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} \cdot \frac{b^\alpha c^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}$$

$$= \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} \cdot \mathrm{Cap}_\alpha(bx+c)$$

and, $\mathrm{Cap}_\alpha(bx+c) \le \frac{b\cdot 1+c}{1^\alpha} = b+c.$

$$\Rightarrow b+c \ge \alpha^\alpha(1-\alpha)^{1-\alpha} \cdot \mathrm{Cap}_{\alpha,1-\alpha}(axz+bx+cz+d). \quad \square$$

Lecture 19: Bilinear forms and capacity preservers.

Last time:

Theorem (AOG '17): $p, q \in \mathbb{R}^{(1,\ldots,1)}_{\geq 0}[x_1, \ldots, x_n]$ real stable and $\alpha \in [0,1]^n \Rightarrow$

$$\langle p, q \rangle^{\bar{1}} \geq \alpha^{\alpha} (\bar{1}-\alpha)^{\bar{1}-\alpha} \, \mathrm{Cap}_{\alpha}(p) \, \mathrm{Cap}_{\bar{1}-\alpha}(q)$$

Proof Idea: Reduce to $n=1$ using standard inductive techniques, then use strong Rayleigh inequalities.

Next: Translate to operators? Recall:

$$T[p](x) = \prod_{i=1}^{n} (\partial_{y_i} + \partial_{z_i})\Big|_{y_i = z_i = 0} \left[\mathrm{Symb}^{\bar{1}}[T](x,z) \cdot p(y) \right]$$

If p, $\mathrm{Symb}^{\bar{1}}[T]$ have non-neg. coeff and are real stable, $\forall x \geq 0, \alpha \in [0,1]^n$:

$$T[p](x) \geq \alpha^{\alpha} (1-\alpha)^{1-\alpha} \, \mathrm{Cap}_{1-\alpha}(\mathrm{Symb}^{\bar{1}}[T](x, \cdot)) \cdot \mathrm{Cap}_{\alpha}(p)$$

Divide by x^{β} and take infs:

Theorem (GL'18): Let $T: \mathbb{R}^{\mathbb{1}}_{\geq 0}[x] \to \mathbb{R}_{\geq 0}[x]$ have real stable symbol.

Then, $\forall$ real stable $p \in \mathbb{R}^{\mathbb{1}}_{\geq 0}[x]$, $\alpha, \beta \in \mathbb{R}^n_{\geq 0}$:

$$\frac{\mathrm{Cap}_\beta(T[p])}{\mathrm{Cap}_\alpha(p)} \geq \alpha^\alpha (1-\alpha)^{1-\alpha} \cdot \mathrm{Cap}_{\beta, 1-\alpha}(\mathrm{Symb}^{\mathbb{1}}[T]).$$

Note: $\langle \cdot, \cdot \rangle^\lambda$ is the bilinear form associated with BB characterization.

$\hookrightarrow$ e.g. $n=1 \Rightarrow \langle x_1^{\kappa_1}, x_1^{\lambda_1-\kappa_1} \rangle^{\lambda_1} = \binom{\lambda_1}{\kappa_1}^{-1}$

and $\langle \mathrm{Pol}^{\lambda_1}(x_1^{\kappa_1}), \mathrm{Pol}^{\lambda_1}(x_1^{\lambda_1-\kappa_1}) \rangle^{(1,\ldots,1)}$

$$= \binom{\lambda_1}{\kappa_1}^{-2} \left\langle \sum_{S \in \binom{[\lambda_1]}{\kappa_1}} x^S, \sum_{T \in \binom{[\lambda_1]}{\kappa_1}} x^{T^c} \right\rangle = \binom{\lambda_1}{\kappa_1}^{-1}$$

$$\Rightarrow \langle p, q \rangle^\lambda = \langle \mathrm{Pol}^\lambda(p), \mathrm{Pol}^\lambda(q) \rangle^{(1,\ldots,1)}$$

Next: Polarization trick used for BB characterization.

Theorem (GL '18): If $p, q \in \mathbb{R}^{\lambda}_{\geq 0}[x_1, \ldots, x_n]$ are real stable, then $\forall \alpha \in \mathbb{R}^n_{\geq 0}$,

$$\langle p, q \rangle^{\lambda} \geq \frac{\alpha^{\alpha}(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^{\lambda}} \mathrm{Cap}_{\alpha}(p) \cdot \mathrm{Cap}_{\lambda-\alpha}(q).$$

Proof: Define $\beta \in \mathbb{R}^{\lambda_1 + \cdots + \lambda_n}_{\geq 0}$ s.t.

$$\beta = \left(\frac{\alpha_1}{\lambda_1} \cdot \bar{1}, \frac{\alpha_2}{\lambda_2} \cdot \bar{1}, \ldots, \frac{\alpha_n}{\lambda_n} \cdot \bar{1}\right).$$

Thus, $\langle p, q \rangle^{\lambda} = \langle \mathrm{Pol}^{\lambda}(p), \mathrm{Pol}^{\lambda}(q) \rangle^{\bar{1}}$

$$\geq \beta^{\beta}(\bar{1}-\beta)^{\bar{1}-\beta} \cdot \mathrm{Cap}_{\beta}(\mathrm{Pol}^{\lambda}(p)) \mathrm{Cap}_{\bar{1}-\beta}(\mathrm{Pol}^{\lambda}(q))$$

$$= \beta^{\beta}(\bar{1}-\beta)^{\bar{1}-\beta} \cdot \mathrm{Cap}_{\alpha}(p) \cdot \mathrm{Cap}_{\lambda-\alpha}(q)$$

by the "symmetriz capacity lemma".

Finally,

$$\beta^{\beta}(\bar{1}-\beta)^{\bar{1}-\beta} = \prod_{i=1}^{n} \left[\left(\frac{\alpha_i}{\lambda_i}\right)^{\frac{\alpha_i}{\lambda_i}}\right]^{\lambda_i} \cdot \left[\left(1-\frac{\alpha_i}{\lambda_i}\right)^{1-\frac{\alpha_i}{\lambda_i}}\right]^{\lambda_i}$$

$$= \prod_{i=1}^{n} \frac{\alpha_i^{\alpha_i}}{\lambda_i^{\alpha_i}} \cdot \frac{(\lambda_i - \alpha_i)^{\lambda_i - \alpha_i}}{\lambda_i^{\lambda_i - \alpha_i}}$$

$$= \frac{\alpha^{\alpha}(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^{\lambda}}. \quad \square$$

Final step: General BB symbol.

Recall: $\mathrm{Symb}^\lambda[T](x,z)$
$$= \sum_{0\le\kappa\le\lambda} \prod_{i=1}^n \binom{\lambda_i}{\kappa_i} \cdot T[x^\kappa] z^{\lambda-\kappa}$$

Thus, $\langle \mathrm{Symb}^\lambda[T](x,z), p(z)\rangle^\lambda$ (in z)
$$= \sum_{0\le\kappa\le\lambda} \prod_{i=1}^n \binom{\lambda_i}{\kappa_i}^{-1} p_\kappa \cdot \prod_{i=1}^n \binom{\lambda_i}{\kappa_i} T[x^\kappa]$$
$$= \sum_{0\le\kappa\le\lambda} p_\kappa \cdot T[x^\kappa] = T[p](x).$$

That is, $T[p](x) = \langle \mathrm{Symb}^\lambda[T](x,z), p(z)\rangle^\lambda$.

Theorem (GL'18): Let $T: \mathbb{R}^\lambda_{\ge0}[x] \to \mathbb{R}_{\ge0}[x]$ have real stable symbol. Then $\forall$ real stable $p \in \mathbb{R}^\lambda_{\ge0}[x]$ and $\alpha,\beta \in \mathbb{R}^n_{\ge0}$:
$$\frac{\mathrm{Cap}_\beta(T[p])}{\mathrm{Cap}_\alpha(p)} \ge \frac{\alpha^\alpha(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \mathrm{Cap}_{\beta,\lambda-\alpha}(\mathrm{Symb}^\lambda[T]).$$

(Proof is the same as multiaffine case: Apply bound to $T[p](x)$ for fixed $x>0$, then divide x^β and limit.)

Proof: $T[p](x) = \langle \mathrm{Symb}^\lambda[T](x,z), p(z)\rangle^\lambda$

$\geq \frac{\alpha^\alpha(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \cdot \mathrm{Cap}_{\lambda-\alpha}(\mathrm{Symb}^\lambda[T](x,\cdot)) \cdot \mathrm{Cap}_\alpha(p)$

Divide by x^β and take infs. $\square$

Also: Degree-agnostic bounds:

$\forall$ real stable $p, q \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$,

$\langle p, q\rangle^\infty := p(\partial_x) q(x)\big|_{x=0} \geq \alpha^\alpha e^{-\alpha} \mathrm{Cap}_\alpha(p) \cdot \mathrm{Cap}_\alpha(q)$

↳ actual inner product

Other bilinear forms can become inner products via

$$q \mapsto x^\lambda \cdot q(x_1^{-1}, \dots, x_n^{-1}).$$

↳ can be turned into linear operator capacity bounds also.

Application: Imperfect matchings of biregular bipartite graphs.

G (m,n)-bipartite, (a,b)-regular $(am = bn)$

Goal: Bound size-k matchings $\mu_k(G)$.

Recall: For $k \le \min(m,n)$,

$$\mu_k(G) = \sum_{S \in \binom{[n]}{k}} \partial_x^S p_G(x)\Big|_{x=\bar{1}} \cdot a^{-(m-k)}$$

where $p_G(x) = \prod_{i=1}^{m} \sum_{j=1}^{n} M_{ij} x_j$,

where M is bipartite adjacency matrix of G.

Consider $T[p] := \left(\sum_{S \in \binom{[n]}{k}} \partial_x^S \Big|_{x=\bar{1}} \right) p$

Note that $p_G(x) \in \mathbb{R}_{\ge 0}^{b \cdot \bar{1}}[x_1, \dots, x_n]$,

so $\mathrm{Symb}^{b \cdot \bar{1}}[T] = T\left[\prod_{i=1}^{n} (x_i + z_i)^b \right]$

$$= \sum_{S \in \binom{[n]}{k}} b^k (z+1)^{(b-1) \cdot S} (z+1)^{b \cdot S^c}$$

$$= b^k (z+1)^{(b-1) \cdot \bar{1}} \cdot \sum_{S \in \binom{[n]}{n-k}} (z+1)^S$$

$$= b^k (z+1)^{(b-1) \cdot \bar{1}} \cdot e_{n-k}(z_1+1, \dots, z_n+1)$$

$\Rightarrow$ real stable, non-neg. coeff.

Recall:

$$\frac{\mathrm{Cap}_\beta(T[p])}{\mathrm{Cap}_\alpha(p)} \ge \frac{\alpha^\alpha (\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \cdot \mathrm{Cap}_{\beta, \lambda-\alpha}(\mathrm{Symb}^\lambda[T](x,z))$$

(Since $\mathrm{Symb}^\lambda[T]$ does not depend on x, we must choose $\beta = 0 \to$ "real stable functional")

Note that $\nabla \log p_G(e^x)\big|_{x=0}$ is given by:

$$\partial_{x_k}\Big|_{x_k=0} \log \prod_{i=1}^m \sum_{j=1}^n m_{ij} e^{x_j} = \partial_{x_k}\Big|_{x_k=0} \sum_{i=1}^m \log \sum_{j=1}^n m_{ij} e^{x_j}$$

$$= \sum_{i=1}^m \frac{m_{ik}}{\sum_{j=1}^n m_{ij}} = \frac{1}{a} \sum_{i=1}^m m_{ik} = \frac{b}{a}.$$

Thus for $\alpha = \frac{b}{a} \cdot \mathbb{1}$,

$$\mathrm{Cap}_\alpha(p_G) = p_G(\mathbb{1}) = a^m$$

(Other degrees $\Rightarrow$ other α's)

Therefore,

(not a coeff!)

$$\frac{T[p_G]}{\mathrm{Cap}_\alpha(p)} \geq \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \underbrace{\mathrm{Cap}_{\bar{0}, b \cdot \mathbb{1} - \alpha}(\mathrm{Symb}[T])}$$

$$\Rightarrow \frac{a^{m-k} \cdot \mu_k(G)}{a^m} \geq \frac{\left(\frac{b}{a}\right)^{n\left(\frac{b}{a}\right)} \left(b - \frac{b}{a}\right)^{n\left(b - \frac{b}{a}\right)}}{b^{nb}} \cdot (\cdots)$$

(Just need to compute Cap)

$$\mathrm{Cap}_{\bar{D},\, b\cdot\bar{1}-\alpha}(\mathrm{Symb}[T])$$

$$= \inf_{X,Z>0} \frac{b^k (z+1)^{(b-1)\cdot\bar{1}} e_{n-k}(z_1+1,\dots,z_n+1)}{z^{(b-\frac{b}{a})\cdot\bar{1}}} \qquad \text{(symmetrize)}$$

Symm. Cap. lemma

$$= \inf_{t>0} \frac{b^k (t+1)^{n(b-1)} \binom{n}{k} (t+1)^{n-k}}{t^{n(b-\frac{b}{a})}}$$

$$= b^k \binom{n}{k} \cdot \mathrm{Cap}_{n(b-\frac{b}{a})}\left((t+1)^{nb-k}\right)$$

fave calc. problem

$$= b^k \binom{n}{k} \cdot \left(\frac{(nb-k)}{n(b-\frac{b}{a})}\right)^{n(b-\frac{b}{a})} \cdot \left(\frac{(nb-k)}{n\cdot\frac{b}{a}-k}\right)^{n\cdot\frac{b}{a}-k}$$

Recall $nb = ma$, to get:

$$\mu_k(G) \geq (ab)^k \binom{n}{k} \cdot \frac{m^m (nb-m)^{nb-m}}{(nb)^{nb}} \cdot \frac{(nb-k)^{nb-k}}{(nb-m)^{nb-m} (m-k)^{m-k}}$$

$$= (ab)^k \binom{n}{k} \frac{m^m (ma-k)^{ma-k}}{(ma)^{ma} (m-k)^{m-k}}.$$

(originally proven by Csikvári using graph theory/entropy methods; pro/con? → our proof required no graph intuition.)

Lecture 20: Applications of Capacity preservers

Last time: Theorem: Given $p, q \in \mathbb{R}^{\lambda}_{\geq 0}[x_1, \ldots, x_n]$, $T: \mathbb{R}^{\lambda}_{\geq 0}[x] \to \mathbb{R}_{\geq 0}[x]$, if $p, q, \mathrm{Symb}^{\lambda}[T]$ are real stable, then

$$\frac{\mathrm{Cap}_{\beta}(T[p])}{\mathrm{Cap}_{\alpha}(p)} \geq \frac{\alpha^{\alpha}(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^{\lambda}} \mathrm{Cap}_{\beta,\lambda-\alpha}(\mathrm{Symb}^{\lambda}[T])$$

and

$$\langle p, q \rangle^{\lambda} \geq \frac{\alpha^{\alpha}(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^{\lambda}} \mathrm{Cap}_{\alpha}(p) \cdot \mathrm{Cap}_{\lambda-\alpha}(q).$$

Application: Imperfect matchings

Recall: $a^{m-k} \cdot \mu_k(G) = \sum_{S \in \binom{[n]}{k}} \partial_x^S p_G(x)\Big|_{x=\bar{1}}$

where $p_G(x) = \prod_{i=1}^{m} \sum_{j=1}^{n} M_{ij} x_j$,

where M is bipartite adjacency matrix of G.

$p_G \in \mathbb{R}^{b\cdot\bar{1}}_{\geq 0}[x_1, \dots, x_n]$, $T(p) = \left(\sum_{S \in \binom{[n]}{k}} \partial_x^S \Big|_{x=\bar{1}} \right) p(x)$

$\mathrm{Symb}^{b\cdot\bar{1}}[T](x,z)$

$$= b^k (z+1)^{(b-1)\cdot\bar{1}} \cdot e_{n-k}(z_1+1, \dots, z_n+1)$$

$\Rightarrow$ real stable, non-neg. coeff.

Thus,

$$\frac{\mathrm{Cap}_\beta(T[p_G])}{\mathrm{Cap}_\alpha(p_G)} \geq \frac{\alpha^\alpha (\bar{1}\cdot b - \alpha)^{\bar{1}\cdot b - \alpha}}{(\bar{1}\cdot b)^{\bar{1}\cdot b}} \mathrm{Cap}_{\beta, \bar{1}\cdot b - \alpha}(\mathrm{Symb}^{b\cdot\bar{1}}[T])$$

T functional $\Rightarrow \beta = 0$

$\nabla \log p_G(e^x)\big|_{x=\bar{0}} = \frac{b}{a}\cdot\bar{1} \Rightarrow \alpha = \frac{b}{a}\cdot\bar{1}$

$\Rightarrow \mathrm{Cap}_\alpha(p_G) = p_G(\bar{1}) = a^m$

Thus,

$$a^{-k}\mu_k(G) \geq \frac{\left(\frac{b}{a}\right)^{n\cdot\frac{b}{a}} \left(b - \frac{b}{a}\right)^{n(b-\frac{b}{a})}}{b^{nb}} \cdot \mathrm{Cap}_{0, b\cdot\bar{1}-\alpha}(\mathrm{Symb})$$

<u>Finally</u>: Compute $\mathrm{Cap}(\mathrm{Symb})$

$$= \mathrm{Cap}_{b\cdot\bar{1}-\alpha}\left(b^k \prod_{i=1}^n (z_i+1)^{b-1} \cdot e_{n-k}(z_1+1, \dots, z_n+1) \right)$$

Note that the polynomial is symm. in z_i, and $b\cdot\bar{1}-\alpha = \left(b-\frac{b}{a}\right)\cdot\bar{1}$.

By Capacity symmetry lemma,

$$= \mathrm{Cap}_{n(b-\frac{b}{a})}\left(b^k (t+1)^{n(b-1)} \binom{n}{k} (t+1)^{n-k}\right)$$

$$= \mathrm{Cap}_{n(b-\frac{b}{a})}\left(b^k \binom{n}{k}\cdot (t+1)^{nb-k}\right)$$

$$= b^k \binom{n}{k}\cdot \mathrm{Cap}_{n(b-\frac{b}{a})}\left((t+1)^{nb-k}\right)$$

fave Calc. probl.

$$= b^k\binom{n}{k}\cdot\left(\frac{nb-k}{n(b-\frac{b}{a})}\right)^{n(b-\frac{b}{a})}\cdot\left(\frac{nb-k}{n\cdot\frac{b}{a}-k}\right)^{n\cdot\frac{b}{a}-k}$$

$$\Rightarrow \mu_k(G) \geq (ab)^k\binom{n}{k}\frac{m^m (ma-k)^{ma-k}}{(ma)^{ma}(m-k)^{m-k}}$$

(Best known lower bound on $\mu_k(G)$, due to Csikvári.)

<u>Also</u>: Need some sort of graph regularity, but some assumptions can be massaged to generalize the bounds/approximations.

Application: Intersection of two matroids

<u>Recall</u>: A <u>matroid</u> M on ground set $\{1,\dots,n\}$ is a non-empty collection of <u>bases</u> $B\subseteq[n]$ s.t. $|B|=d$ $\forall B\in M$ and

(Exch) $\forall B_1,B_2\in M$, $\forall i\in B_1\setminus B_2$, $\exists j\in B_2\setminus B_1$ s.t. $B_1\setminus\{i\}\cup\{j\}\in M$.

Basis-gen. polyn.

$$p_M(x)=\sum_{B\in M}x^B \text{ is Lorentzian.}$$

<u>Matroid intersection problem</u>:

Given M_1, M_2 on same ground set $[n]$, count $|M_1\cap M_2|$.

<u>One idea</u>: $f=p_{M_1}$, $g=p_{M_2}$, $\tilde g=x^{\mathbb{1}}\cdot g(x^{-1})$

$$|M_1\cap M_2|=\sum_{S\subseteq[n]}f_S g_S=\sum_{S\subseteq[n]}f_S\tilde g_{S^c}$$

$$=\langle f,\tilde g\rangle^{\mathbb{1}}\overset{?}{\geq}\alpha^\alpha(\mathbb{1}-\alpha)^{\mathbb{1}-\alpha}\cdot \mathrm{Cap}_\alpha(f)\cdot \mathrm{Cap}_{\mathbb{1}-\alpha}(\tilde g)$$

$$= \alpha^{\alpha}(\bar{1}-\alpha)^{\bar{1}-\alpha}\,\mathrm{Cap}_\alpha(f)\cdot \inf_{x>0} \frac{x^{\bar{1}}\cdot g(x^{-1})}{x^{\bar{1}-\alpha}}$$

$$= \alpha^{\alpha}(\bar{1}-\alpha)^{\bar{1}-\alpha}\,\mathrm{Cap}_\alpha(f)\cdot \mathrm{Cap}_\alpha(g).$$

Problems:

1) $f, \tilde{g}$ not real stable

↳ we will assume real stable, but similar arguments work in general

2) $g \mapsto \tilde{g}$ does not preserve Lorentzian!

↳ it does for matroid basis gen. polys. → $\tilde{g}$ corresp. to dual matroid

3) $f(x) = p_{M_1}(x)$ might be hard to evaluate. ($p_{M_1}(\bar{1}) = |M_1|$) → how to compute $\mathrm{Cap}_\alpha(f)$?

↳ we will avoid this using some entropy bounds/facts

Let $\mathcal{S}$ be any collection of subsets of $[n]$ (e.g., $\mathcal{S} = M_1 \cap M_2$) and let $\nu_{\mathcal{S}}$ denote the uniform prob. distribution on $\mathcal{S}$. Then:

$$H(\nu_{\mathcal{S}}) = -\sum_{s \in \mathcal{S}} p_s \log p_s = -\sum_{s \in \mathcal{S}} \frac{1}{|\mathcal{S}|} \log\left(\frac{1}{|\mathcal{S}|}\right)$$

$$= \log |\mathcal{S}|$$

Thus we can approximate $|\mathcal{S}|$ by approximating the entropy of $\nu_{\mathcal{S}}$.

<u>Fact (entropy subadditivity)</u>:

If ν is a distribution on $2^{[n]}$ and γ are the marginals of ν

$\left(\gamma_i = \mathbb{P}_{S \sim \nu}[i \in S]\right)$, then

$$H(\nu) \leq \sum_{i=1}^{n} H(\mathrm{Ber}(\gamma_i)).$$

<u>Proof</u>: Let μ be the product distribution of $2^{[n]}$ corresponding

to γ. That is,

$$\mu_S = \prod_{i\in S} \gamma_i \prod_{i\notin S} (1-\gamma_i).$$

(Note that $\sum_{S\subseteq[n]} \mu_S$

$$= \sum_{S\subseteq[n]} \gamma^S (\bar{1}-\gamma)^{S^c} = \prod_{i=1}^{n} (\gamma_i + (1-\gamma_i)) = 1.)$$

Now, $0 \le D_{KL}(\nu\|\mu) = \sum_{S\subseteq[n]} \nu_S \cdot \log\left(\frac{\nu_S}{\mu_S}\right)$

$$= -H(\nu) - \sum_{S\subseteq[n]} \nu_S \left(\sum_{i\in S} \log(\gamma_i) + \sum_{i\notin S} \log(1-\gamma_i)\right)$$

$$= -H(\nu) - \sum_{i=1}^{n} \log(\gamma_i) \cdot \sum_{S\ni i} \nu_S$$

$$- \sum_{i=1}^{n} \log(1-\gamma_i) \sum_{S\not\ni i} \nu_S$$

Now, $\sum_{S\ni i} \nu_S = \mathbb{P}_{S\sim\nu}[i\in S] = \gamma_i$

$$\sum_{S\not\ni i} \nu_S = \mathbb{P}_{S\sim\nu}[i\notin S] = 1-\gamma_i$$

$$\Rightarrow = -H(\nu) - \sum_{i=1}^{n} \left[\gamma_i \log \gamma_i + (1-\gamma_i)\log(1-\gamma_i)\right]$$

$$= -H(\nu) + \sum_{i=1}^{n} H(\mathrm{Ber}(\gamma_i)). \quad \square$$

Lecture 21: Matroid intersection

Recall: $p_M(x) = \sum_{B \in M} x^B$

is Lorentzian, (but we assume here real stable.)

Define $f(x) = p_{M_1}(x)$, $g(x) = p_{M_2}(x)$,

$\tilde{g}(x) = x^{\mathbb{1}} g(x^{-1})$, and

$$|M_1 \cap M_2| = \langle f, \tilde{g} \rangle^{\mathbb{1}} \geq \alpha^{\alpha}(\mathbb{1}-\alpha)^{\mathbb{1}-\alpha} \cdot \mathrm{Cap}_{\alpha}(f) \cdot \mathrm{Cap}_{\alpha}(g).$$

For any S, collection of subsets of $[n]$, if ν is unif. distr., then

$$H(\nu) = -\sum_S \nu_S \log \nu_S = \log|S|$$

<u>Lemma (subadditivity)</u>: If ν is distr. on $2^{[n]}$ w/ marginals γ, then

$$H(\nu) \leq \sum_{i=1}^n H(\mathrm{Ber}(\gamma_i)).$$

Now: $S = M_1 \cap M_2 \Rightarrow$

$$H(\nu) = \log|M_1 \cap M_2| \Rightarrow$$

$$\sum_{i=1}^{n} H(Ber(\gamma_i)) \geq H(\nu)$$

$$= \log(|M_1 \cap M_2|) = \log \langle f, \tilde{g} \rangle^{I}$$

$$\geq - \sum_{i=1}^{n} H(Ber(\alpha_i))$$

$$+ \log Cap_\alpha(f) + \log Cap_\alpha(g).$$

That is, $\forall \alpha \in [0,1]^n$ and γ the marginals of ν, we have

$$\sum_{i=1}^{n} H(Ber(\gamma_i)) \geq \log|M_1 \cap M_2|$$

$$\geq - \sum_{i=1}^{n} H(Ber(\alpha_i)) + \log Cap_\alpha(f) + \log Cap_\alpha(g).$$

Problems:

1) γ can be just as hard to compute as $|M_1 \cap M_2|$

2) Need to deal with capacity terms.

Lemma (log-concave superadditivity, AOV '18): If f is the probability generating function of distr. ν with marginals γ, and f is log-concave in $\mathbb{R}^n_{>0}$, then $H(\nu) \geq -\sum_{i=1}^n \gamma_i \log \gamma_i$.

Proof: Let X denote a random variable on $\mathbb{R}^n$ given by 1_S, where $S \sim \nu$, so that $\mathbb{E}[X] = \gamma$.

By log-concavity of f, we have that $-\log f\left(\frac{x_1}{\gamma_1}, \dots, \frac{x_n}{\gamma_n}\right)$ is convex. Thus by Jensen's inequality,

$$0 = -\log f\left(\frac{\mathbb{E}[X]}{\gamma}\right) \leq \mathbb{E}\left[-\log f\left(\frac{X}{\gamma}\right)\right]$$

$$= \sum_S f_S\left[-\log f\left(\frac{1_S}{\gamma}\right)\right]$$

$$= -\sum_S f_S \cdot \log\left[\sum_{T \subseteq S} f_T \cdot \gamma^{-T}\right] \qquad \text{(log monotone increasing)}$$

$$\leq -\sum_S f_S \log\left(f_S \cdot \gamma^{-S}\right)$$

$$= -\sum_S f_S \log f_S + \sum_S f_S \log(\gamma^S)$$

$$= H(\gamma) + \sum_S f_S \sum_{i \in S} \log(\gamma_i)$$

$$= H(\gamma) + \sum_{i=1}^{n} \log(\gamma_i) \cdot \sum_{S \ni i} f_S$$

$$= H(\gamma) + \sum_{i=1}^{n} \gamma_i \log(\gamma_i). \qquad \square$$

Now recall: Fix $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$, $p(\mathbb{1}) = 1$, let γ be the corresponding distribution, and α in the rel. int. of the Newton polytope of p. Then:

$$\log \mathrm{Cap}_\alpha(p) = - \underset{\mathbb{E}[\mu] = \alpha}{\overset{(\inf)}{\min}} D_{KL}(\mu \| \gamma)$$

If all coeff. of $f = c \cdot p$ equal 1, then

$$\log \mathrm{Cap}_\alpha(f) = \log|S| - D_{KL}(\mu^* \| \gamma)$$

↑ still prob. distr.

$$= \log|S| - \sum_S \mu^*_S \log\left(\frac{\mu^*_S}{|S|^{-1}}\right)$$

$$= - \sum_S \mu^*_S \log \mu^*_S = H(\mu^*).$$

And finally, if p is a log-concave function, then the prob. gen. function of μ^* is equal to $p(w \circ x)$ for some $w \in \mathbb{R}^n_{>0}$ (really in $[0,\infty]^n$), which is also log-concave. And, to optimize capacity, we must have $\nabla \log f(e^x)\big|_{x=\vec{0}} = \alpha$. Thus:

$$\boxed{\text{$\log \mathrm{Cap}_\alpha(f)$ is the entropy of a log-concave distribution with marginals α}}$$

<u>Therefore</u>: For any α

$$\sum_{i=1}^n H(\mathrm{Ber}(\gamma_i)) \geq \log |M_1 \cap M_2| \geq$$
$$-\sum_{i=1}^n H(\mathrm{Ber}(\alpha_i)) + \log \mathrm{Cap}_\alpha f + \log \mathrm{Cap}_\alpha g$$
$$\geq -\sum_{i=1}^n H(\mathrm{Ber}(\alpha_i)) - 2\sum_{i=1}^n \alpha_i \log \alpha_i$$
$$= \sum_{i=1}^n H(\mathrm{Ber}(\alpha_i)) + 2\sum_{i=1}^n (1-\alpha_i)\log(1-\alpha_i).$$

Also, $\phi(t) = t + (1-t)\log(1-t)$

$\Rightarrow \phi'(t) = 1 - \log(1-t) - 1 = -\log(1-t)$

and $\phi(0) = 0 \to \phi(t) \geq 0$.

Thus,

$$2\sum_{i=1}^{n}(1-\alpha_i)\log(1-\alpha_i) \geq -2\sum_{i=1}^{n}\alpha_i = -2d,$$

where d is the rank of M_1, M_2 / the degree of f, g.

<u>Combining everything gives:</u>

$$\sum_{i=1}^{n} H(\mathrm{Ber}(\gamma_i)) \geq \log|M_1 \cap M_2|$$
$$\geq \sum_{i=1}^{n} H(\mathrm{Ber}(\alpha_i)) - 2d \quad \text{for all } \alpha.$$

<u>Problem remains</u>: How to deal with γ?

<u>Answer</u>: Optimization.

<u>Theorem (AOG '18)</u>: Given two matroids M_1, M_2 of rank d,

$L^* \geq \log|M_1 \cap M_2| \geq L^* - 2d$, where

$$L^* := \sup_{\alpha \in hull(M_1) \cap hull(M_2)} \underbrace{\sum_{i=1}^{n} H(Ber(\alpha_i))}$$

↳ concave function.

Further, L^* can be computed/ approximated efficiently (given indep. oracle).

<u>Proof</u>: The only thing left to prove is that we can compute L^* efficiently. We use the ellipsoid method, for which we need a separation oracle for $hull(M_1) \cap hull(M_2)$, which is implied by a separation oracle for $hull(M_1)$ and $hull(M_2)$ individually. Note that matroid basis polytopes ($hull(M_i)$) characterize collections of subsets for which Kruskal's greedy algo can be used to maximize linear

functionals, given an independence oracle. Efficient optimization of linear functionals over hull(M_i) imply an efficient separation oracle over hull(M_i). This completes the proof, up to many details left out. $\square$