Lecture 8 – Lorentzian Characterization theorem

Last time:

Definition: A d-homogeneous polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is **Lorentzian** if:

(P) The coefficients of p are nonnegative

(Q) $\forall v_1, \ldots, v_{d-2}$, the Hessian of $D_{v_1} \cdots D_{v_{d-2}} p$ has at most one positive eigenvalue.

(Unweildy/non-combinatorial defn, so we discussed the following characterization theorem)

Theorem: A d-homogeneous polynomial $p \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is Lorentzian if and only if:

1) $\forall \mu \in \mathbb{Z}^n_{\geq 0}$, $|\mu| \leq d-2$, $\partial_x^\mu p$ is indecomposable (cannot be written as the sum of two polynomials on disjoint sets of variables)
2) $\forall \mu \in \mathbb{Z}^n_{\geq 0}$, $|\mu| = d-2$, the Hessian of $\partial_x^\mu p$ has at most one positive eigenvalue.

(Much more combinatorial conditions, since partial derivatives often correspond to comb. ops.)

Corollary 1: A d-homogeneous polyn. $p \in \mathbb{R}_{\geq 0}[x, y]$ is Lorentzian iff:

1) Its support has no "holes"
2) Its coeff. satisfy Newton's inequalities.

Given a matroid M, let
$p_M(x) = \sum_{B \in M} x^B$ be its basis generating polynomial.

<u>Corollary 2</u>: We have that p_M is Lorentzian for all matroids M iff:

1) p_M is indecomposable for all matroids M of rank ≥ 2.
2) The Hessian of p_M has at most 1 pos. eval. for all matroids M of rank $= 2$.

(We will prove this later.)
(Upshot: fills the "gaps" left by the real stability theory)

<u>Goal today</u>: Prove the characterization theorem.

(side board)

Lemma: If $g \in \mathbb{R}[x_1, \dots, x_n]$ is d-homogeneous ($d \geq 2$) and $g(x) > 0$, then the following are equivalent:

(a) the Hessian of g at x has exactly one pos. eval

(b) the Hessian of $g^{1/d}$ is negative semidef. at x

(c) the matrix
$$d \cdot g \cdot \nabla^2 g - (d-1) \cdot \nabla g (\nabla g)^T$$
is negative semidefinite at x

Proof: Exercise.

Lemma (Bochner method): Fix d-homog. $f \in \mathbb{R}[x_1, \dots, x_n]$ with $d \geq 3$ and $x \in \mathbb{R}^n_{>0}$.

If: (1) $\partial_{x_i} f(x) > 0 \ \forall i$,

(2) the Hessian of $\partial_{x_i} f$ at x has exactly one pos. eval, and

(3) the Hessian of f at x is irreducible with (def: side board) non-negative off-diagonal entries,

then the Hessian of f at x has exactly one pos. eval.

<u>Proof</u>: We apply (c) from the previous lemma to $\partial_{x_i} f$ to get

$$(d-1)\,\partial_{x_i} f \cdot \nabla^2 \partial_{x_i} f \preceq (d-2)\cdot \nabla \partial_{x_i} f\,(\nabla \partial_{x_i} f)^T,$$

at x, and since $\partial_{x_i} f(x) > 0$, $x > 0$, we have

$$x_i \cdot \nabla^2 \partial_{x_i} f \preceq \frac{d-2}{d-1} \cdot \frac{x_i}{\partial_{x_i} f} \cdot \nabla \partial_{x_i} f\,(\nabla \partial_{x_i} f)^T$$

$\forall i$, at x. Now apply Euler's identity (side board) $d \cdot f(x) = \sum_{i=1}^{n} x_i \cdot \partial_{x_i} f(x)$

to the entries of $\nabla^2 f$ to get:

$$(d-2)\,\nabla^2 f = \sum_{i=1}^{n} x_i\, \nabla^2 \partial_{x_i} f,$$ and

combining the above gives

$$(d-2)\,\nabla^2 f \preceq \frac{d-2}{d-1} \sum_{i=1}^{n} \frac{x_i}{\partial_{x_i} f} \cdot \nabla \partial_{x_i} f\,(\nabla \partial_{x_i} f)^T.$$

Finally this implies

$$(d-1)\nabla^2 f \preceq (\nabla^2 f)\Lambda(\nabla^2 f),$$

where $\Lambda = \mathrm{diag}\left(\frac{x_1}{\partial_{x_1} f}, \dots, \frac{x_n}{\partial_{x_n} f}\right)$.

(Side board: write out this expression to demonstrate.)

For $B = \Lambda^{1/2}(\nabla^2 f)\Lambda^{1/2}$,

this implies $B^2 - (d-1)B \succeq 0$.

$\Rightarrow$ No eigenvalue of B in $(0, d-1)$.

Side board: <u>Perron-Frobenius Theorem</u>:
An irreducible matrix with non-negative off-diagonal entries has a simple real eigenvalue with uniquely maximum real part, and the corresponding eigenvector has positive entries.

Note that $B(\Lambda^{-1/2}x) =$

$$\Lambda^{1/2}(\nabla^2 f)x =$$

$$\Lambda^{1/2}\begin{bmatrix} \partial_{x_1}\partial_{x_1} f & \partial_{x_2}\partial_{x_1} f & \cdots & \partial_{x_n}\partial_{x_1} f \\ \vdots & \vdots & & \vdots \\ \partial_{x_1}\partial_{x_n} f & \partial_{x_2}\partial_{x_n} f & \cdots & \partial_{x_n}\partial_{x_n} f \end{bmatrix} x = \Lambda^{1/2}\begin{bmatrix} (d-1)\partial_{x_1} f \\ \vdots \\ (d-1)\partial_{x_n} f \end{bmatrix}$$

by Euler's identity.

Thus, $B(\Delta^{-1/2}x) = (d-1)\Delta^{-1/2}x$

$\Rightarrow$ $d-1$ is eigenvalue of B with positive eigenvector.

Thus by the Perron-Frobenius theorem, B has exactly one positive eigenvalue, and thus so does $D^2 f(x)$. $\square$

Lecture 9: Lorentzian characterization and further properties

Last time:

Lemma (Bochner method): Fix d-homog. $f \in \mathbb{R}[x_1, \dots, x_n]$ with $d \geq 3$ and $v \in \mathbb{R}^n_{>0}$.

If: (1) $\partial_{x_i} f(v) > 0$ $\forall i$,

(2) the Hessian of $\partial_{x_i} f$ at v has exactly one pos. eval,

(3) the Hessian of f at v is irreducible with non-negative off-diagonal entries,

then the Hessian of f at v has exactly one pos. eval.

Proof: Opaque-ish proof leading to two conclusions for some assoc. matrix B: No e-vals in $(0, d-1)$, and no e-vals $> d-1$.

Lemma: If $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is d-homog with $d \geq 3$ and f is indecomposable, then so is $D_v f$, $\forall v \in \mathbb{R}^n_{>0}$.

Proof: β $D_v f = g + h$, where g, h depend on disjoint sets of variables, S and S^c. Since f is indecomposable, there is some present monomial $x^\alpha x^\beta$, where x^α, x^β depend on variables in S, S^c resp. Since $d \geq 3$, WLOG $|\alpha| \geq 2$. Thus for some i, $\partial_{x_i} f$ depends on variables in S and S^c, a contradiction. □

Theorem: Let $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ be d-homogeneous with $d \geq 3$. If (1) f is indecomposable, and (2) $\partial_{x_i} f$ is Lorentzian, $\forall i$, then f is Lorentzian.

Proof: If $d \geq 4$, then $\forall v \in \mathbb{R}^n_{>0}$, $D_v f$ is indecomposable and $\partial_{x_i} D_v f = D_v \partial_{x_i} f$ is Lorentzian $\forall i$ by defn. of Lorentzian. Thus by induction $D_v f$ is Lorentzian $\forall v \in \mathbb{R}^n_{>0}$, which implies f is Lorentzian by definition.

If $d = 3$, then for $v \in \mathbb{R}^n_{>0}$, $\nabla^2 f(v) \cong D_v(\nabla^2 f) = \nabla^2 D_v f$, which is irreducible by indecompos. of f and positivity of v. Thus we can apply the Bochner Lemma to f to get that $\nabla^2 f(v)$ has exactly one positive eval. Since $\nabla^2 f(v) \cong \nabla^2(D_v f)$, this implies $D_v f$ is Lorentzian, and the result follows. $\square$

Theorem: A d-homogeneous polynomial $p \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is Lorentzian if and only if:

1) $\forall \mu \in \mathbb{Z}^n_{\geq 0}$, $|\mu| \leq d-2$, $\partial_x^\mu p$ is indecomposable (cannot be written as the sum of two polynomials on disjoint sets of variables)

2) $\forall \mu \in \mathbb{Z}^n_{\geq 0}$, $|\mu| = d-2$, the Hessian of $\partial_x^\mu p$ has at most one positive eigenvalue.

Proof: ($\Leftarrow$) For $d=2$, immediate.

For $d \geq 3$, $\partial_{x_i} p$ is Lorentzian $\forall i$ by induction, and p is indecomposable by assumption. Thus p is Lorentzian.

NEXT →

($\Rightarrow$) By limiting, $\partial_{x_i} p$ is Lorentzian for all i.
(Consider $D_{v_1} \cdots D_{v_{d-3}} D_{e_i + \varepsilon \mathbb{1}} p$ and let $\varepsilon \downarrow 0$.)
By induction, we only need to show p is indecomposable, and when $d=2$ Hessian p has at most one pos. eval (follows immediately from definition).
Suppose $p = g + h$ where g, h depend on disjoint sets of variables.
Then $D_{\mathbb{1}}^{d-2} p = D_{\mathbb{1}}^{d-2} g + D_{\mathbb{1}}^{d-2} h$ is a sum of nonzero polynomials depending on disjoint sets of variables, and $D_{\mathbb{1}}^{d-2} p$ is also Lorentzian.
Thus up to reordering variables, the Hessian of $D_{\mathbb{1}}^{d-2} p$ can

be written as

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where A, B have non-negative entries and are not identically zero. Since A, B real symm. and $tr(A), tr(B) \geq 0$, each of A and B have at least one pos. eval. Thus Hessian of $D_{I}^{d-2} p$ has at least 2 pos. evals., a contradiction. □

(Corollary that a bivariate homogeneous polynomial is Lorentzian iff its coefficient sequence has no holes and satisfies Newton's inequalities.)

Lecture 10 — Matroid polynomials

Definition: A *matroid* M on a finite ground set E is defined by a non-empty collection of subsets of E all of the same size (the *rank* of M), called *bases* of M, which satisfy:

(Exch) $\forall B_1, B_2 \in M$, $\forall i \in B_1 \setminus B_2$, $\exists j \in B_2 \setminus B_1$ such that $B_1 \cup \{j\} \setminus \{i\} \in M$.

Let $p_M((x_e)_{e\in E}) := \sum_{B \in M} x^B$, called the *basis generating polynomial*.

Theorem: For any matroid M, p_M is Lorentzian.

Recall: We already showed that to prove this, we just need to show that

(1) p_M is indecomposable $\forall M, rk(M) \geq 2$

(2) Hessian of p_M has at most one pos. eval $\forall M, rk(M) = 2$

Proof: (1) Fix M and suppose

$$p_M(x) = f((x_e)_{e \in S}) + g((x_e)_{e \in S^c})$$

$f, g \not\equiv 0$.

Choose $B_1, B_2 \in M$ s.t. $B_1 \subseteq S$, $B_2 \subseteq S^c$. Apply exchange axiom to get $B := B_1 \cup \{f\} \setminus \{e\} \in M$, with $e \in S, f \in S^c$. Since rank ≥ 2, $B \cap S \neq \emptyset$, $B \cap S^c \neq \emptyset$, a contradiction.

(2) What does a rank-2 matroid look like?

By removing loops of M (unused vars.) we may assume every $e \in E$ is contained in some basis of M.

<u>Claim</u>: $\{e, f\} \notin M$ is an equivalence relation on E.

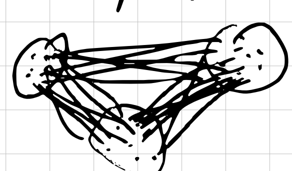
<u>Pf.</u>: If $\{e_1, e_2\} \notin M$, $\{e_2, e_3\} \notin M$

If $\{e_1, e_3\} \in M$, fix any $\{e_2, f\} \in M$ and apply exch. axiom $\Rightarrow$

$\{e_2, f\} \cup \{e_i\} \setminus \{f\} \in M$ for some $i \in \{1, 3\} \Rightarrow$ contradiction. $\square$

Thus, E breaks up into equiv. classes, and for any e, f in distinct classes, $\{e, f\} \in M$.

I.e., complete multipartite graph:



$G = (E, M)$

Now, $p_M(x) = x^T A x$, what does A look like? Order the variables by equivalence class:

$$\begin{array}{c} C_1 \\ C_2 \\ \vdots \\ C_m \end{array} \left[\begin{array}{c|c|c|c} O & J & \cdots & J \\ \hline J & O & \cdots & J \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline J & J & \cdots & O \end{array}\right] \cdot \frac{1}{2} = A$$

$$\begin{array}{cccc} C_1 & C_2 & \cdots & C_m \end{array}$$

J = all-ones matrix

$$A = \frac{1}{2}\left(J_E - J_{C_1} - J_{C_2} - \cdots J_{C_m}\right)$$

$$= (\text{rank-one PSD}) - (\text{PSD})$$

Thus A has at most one positive eigenvalue. $\square$

(In fact this characterizes matroids: Let M be a collection of subsets of E, all of the same cardinality. Then M is the set of bases of a matroid iff $p_M(x) = \sum_{B \in M} x^B$ is Lorentzian. More generally, the support of any multiaffine Lorentzian polynomial is the set of bases of a matroid.)

Mason's conjecture ('70s, BH '19, ALOV '19)

The strongest form of the conjecture says the set of independent sets (subsets of bases) of size k of a matroid form an ultra log-concave sequence (w.r.t. $|E| = n$).

A set $S \subseteq E$ is <u>independent</u> if $S \subseteq B$ for some $B \in M$.

Construct <u>independent set gen. polynomial</u>:

$$q_M((x_e)_{e\in E}, y) = \sum_{\substack{I \subseteq E \\ \text{indep.}}} x^I y^{n-|I|}$$

If q_M is Lorentzian, then $q_M(t, \dots, t, s)$ has coefficients given by the number of size-k indep. sets, and the coeff are ultra log-concave.

$\Rightarrow$ proves Mason's conjecture

(<u>Lemma</u>: Truncation is a matroid.)

<u>Theorem</u>: For any matroid M, q_M is Lorentzian.

<u>Proof</u>: Note that $\partial_{x_e} q_M = q_{M/e}$, as w/ the basis gen. poly.. So, to prove the theorem,

we need to show:

1) $\partial_y^k q_M$ is indecomposable for all matroids M and $k \le n-2$.

2) $\partial_y^{n-2} q_M$ has Hessian with at most one pos. eval for all matroids M.

For (1), the lowest degree term in y has support the set of bases of some matroid (possibly a truncation of some other matroid). Thus (1) follows from the same argument as in the basis gen. poly. case.

For (2), we have

$$\frac{1}{(n-2)!} \partial_y^{n-2} q_M(x,y) = \frac{n(n-1)}{2} y^2 + (n-1) y \sum_{e \in E} x_e + \sum_{\{e,f\} \in M_2} x_e x_f$$

where M_2 is the truncation.

NEXT ↗

basis gen. poly.
↓

If $p_{M_2}(x) = \frac{1}{2} x^T A x$, then

$$\frac{1}{(n-2)!} \partial_y^{n-2} q_M(x) = \frac{1}{2} \begin{bmatrix} \bar{x} \\ y \end{bmatrix}^T B \begin{bmatrix} \bar{x} \\ y \end{bmatrix}, \text{ where}$$

$$B = \begin{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} & \begin{matrix} (n-1) \\ \vdots \\ (n-1) \end{matrix} \\ (n-1) \cdots (n-1) & n(n-1) \end{bmatrix}$$

(By Cauchy interlacing, B has at most 2 positive evals.)

(To prove this has at most one e-val, we need to find an n-dim. space V (codim 1) in $\mathbb{R}^{n+1}$ s.t. $v^T B v \leq 0 \ \forall v \in V$. We claim it is the orth. compl. of the last column of B. Need a Lemma.)

<u>Lemma</u>: $\forall x \in \mathbb{R}^n$, $\frac{1}{n} \sum_{i=1}^n x_i^2 \geq \frac{1}{n^2} \sum_{i,j=1}^n x_i x_j$

$$\Longleftrightarrow \sum_{i=1}^n x_i^2 \geq \frac{1}{n} \left(\sum_{i=1}^n x_i \right)^2.$$

<u>Proof</u>: $\left(\frac{1}{\sqrt{n}} \mathbb{1}\right)^T x x^T \left(\frac{1}{\sqrt{n}} \mathbb{1}\right) \leq \lambda_{max}(xx^T) = tr(xx^T)$.

Now we compute $v^T B v$ for any v s.t. $v \cdot \binom{\bar{1}}{n} = 0$ with $v = \binom{w}{v_0}$. We have

$$v^T B v = w^T A w + 2(n-1) v_0 \cdot (w \cdot \bar{1}) + n(n-1) v_0^2$$

$$= w^T A w - n(n-1) v_0^2$$

Let $A = J - \sum_{i=1}^{m} J_{C_i}$ where C_i are the equiv. classes of parallel elements of M_2.

Then, $w^T A w = (w \cdot \bar{1})^2 - \sum_{i=1}^{m} \left(\sum_{j \in C_i} w_j \right)^2$

$$\leq \frac{m-1}{m} (w \cdot \bar{1})^2 \leq \frac{n-1}{n} (w \cdot \bar{1})^2$$

by the previous Lemma, since $m \leq m$.

We then finally have

$$v^T B v = w^T A w - n(n-1) v_0^2$$

$$\leq \frac{n-1}{n} (w \cdot \bar{1})^2 - n(n-1) v_0^2$$

$$= n(n-1) v_0^2 - n(n-1) v_0^2 = 0.$$

$\square$

Lec. 11: Other properties of Lorentzian polynomials

Last time: Matroid polynomials

$$p_M(x) = \sum_{B \in M} x^B$$

basis gen. polyn.

$$q_M(x,y) = \sum_{\substack{I \subseteq E \\ \text{indep.}}} x^I y^{n-|I|}$$

Indep. set gen. polyn.

Both are Lorentzian for all matroids M.

Mason's Conjecture: If c_k = # of indep sets of size k in M, then c_k is ultra log-concave, with respect to $n = |E|$.

Pf.: $q_M(t \cdot \bar{1}, s) = \sum_{k=0}^{n} c_k t^k s^{n-k}$

$q_M(t \cdot \bar{1}, s)$ Lorentzian implies the result. Why Lorentzian?

Proposition: If $p, q \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ are Lorentzian polynomials, then so are the following:

(a) $D_v p$ for $v \in \mathbb{R}^n_{\geq 0}$ and $p|_{x_i=0}$.

(b) $p(Ax)$ for all $n \times m$ matrices A with non-negative entries

(c) $p(a \cdot t + b \cdot s) \in \mathbb{R}_{\geq 0}[t, s]$ for all $a, b \in \mathbb{R}^n_{\geq 0}$.

(d) $p(x) \cdot q(z) \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n, z_1, \dots, z_n]$.

(e) $p(x) \cdot q(x) \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$.

Proof:

(a) $D_v p$ by limiting, $p|_{x_i=0}$ by Cauchy interlacing theorem.

(b) Compute

$$\partial_{x_j} p(a_{11}x_1 + \dots a_{1m}x_m, \dots, a_{n1}x_1 + \dots + a_{nm}x_m)$$

$$= \sum_{i=1}^{n} a_{ij} (\partial_{x_i} p)(Ax)$$

Thus for $v \in \mathbb{R}^m_{\geq 0}$, $D_v[p(Ax)]$

$$= \sum_{j=1}^{m} v_j \partial_{x_j} [p(Ax)]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \cdot v_j \cdot (\partial_{x_i} p)(Ax)$$

$$= (D_w p)(Ax)$$

for some $w \in \mathbb{R}^n_{\geq 0}$. By (a) and induction, we need to prove it for $d=2$. In this case $p(x) = x^T M x$, which implies $p(Ax) = x^T(A^T M A)x$. The matrix $A^T M A$ has non-negative entries, and at most one pos. eval. (why?).

(c) Follows from (b) with

$$A = \begin{bmatrix} 1 & 1 \\ a & b \\ 1 & 1 \end{bmatrix}.$$

(d) Use characterization theorem.

$\forall \mu \in \mathbb{Z}_{\geq 0}^{2n}$ with $\mu = \alpha + \beta$,

$|\mu| \leq d-2 \Rightarrow$

$$\partial_x^{\alpha} \partial_z^{\beta} [p(x) q(z)] = (\partial_x^{\alpha} p(x))(\partial_z^{\beta} q(z))$$

is Lorentzian by induction, for $|\mu| \geq 1$. So, need to show $p(x) \cdot q(z)$ is indecomposable, and $p(x) \cdot q(z)$ is Lorentzian when $\deg(p(x) \cdot q(z)) = 2$.

If $p(x) \cdot q(z) = g + h$, depending on disjoint sets of variables, then let $x^{\alpha} z^{\beta}$ be a term of g and $x^{\gamma} z^{\delta}$ a term of h. Thus $x^{\alpha} z^{\delta}$ and $x^{\gamma} z^{\beta}$ are terms of $p(x) \cdot q(z)$, a contradiction.

Further, if $\deg(p(x) \cdot q(z)) = 2$, then $p(x) \cdot q(z)$ is obviously Lorentzian unless $\deg(p) = \deg(q) = 1$. In this case,

$$p(x) \cdot q(z) = x^T v w^T z \quad \text{for}$$

$v, w \in \mathbb{R}^n_{\geq 0}$, and thus

$$2 \cdot p(x) \cdot q(z) = \begin{bmatrix} | \\ x \\ | \\ | \\ z \\ | \end{bmatrix}^T \left[\begin{array}{c|c} 0 & v w^T \\ \hline w^T v & 0 \end{array}\right] \begin{bmatrix} | \\ x \\ | \\ | \\ z \\ | \end{bmatrix}$$

This matrix is real symmetric, rank ≤ 2 and traceless

$\Rightarrow$ has at most one pos. eval.

(e) Follows from (b) and (d).

Let $f(x, z) = p(x) \cdot q(z)$, and consider $A = [I_n, I_n] \Rightarrow$

$$f(Ax) = f(x, x) = p(x) \cdot q(x)$$

is Lorentzian. $\square$

(**Upshot**: Similar properties to real stable polynomials, but harder to get!)

(**Corollary**: Given a matroid M and some subset $S \subseteq E$ of the ground set, the number of bases c_k containing exactly k elements of S forms an ultra log-concave sequence (w.r.t. rank).

Proof: Consider $p_M(\underbrace{t, \dots, t}_{e \in S}, \underbrace{s, \dots, s}_{e \notin S})$.)

Corollary: Mason's conjecture.

(Another operator we would like to have: polarization. Note that it preserves homogeniety. With this we could get linear preservers theorem.)

Recall: $\mathrm{Pol}^d[x^k] = \frac{1}{\binom{d}{k}} \sum_{S \in \binom{[d]}{k}} x^S$

Thus, $\partial_{x_d} \mathrm{Pol}^d[x^k]$

$$= \frac{1}{\binom{d}{k}} \sum_{S \in \binom{[d-1]}{k-1}} x^S$$

$$= \frac{k}{d} \left[\frac{1}{\binom{d-1}{k-1}} \sum_{S \in \binom{[d-1]}{k-1}} x^S \right]$$

$$= \frac{k}{d} \mathrm{Pol}^{d-1}[x^{k-1}]$$

$$= \mathrm{Pol}^{d-1}\left[\frac{1}{d} \partial_x (x^k)\right]$$

That is, $\frac{1}{d}\partial_x$ commutes with ∂_{x_d} through polarization operator.

Theorem: Given $p \in \mathbb{R}^{\lambda}_{\geq 0}[x_1, \dots, x_n]$, if p is Lorentzian, then $\mathrm{Pol}^{\lambda}[p]$ is Lorentzian.

Proof: Just need to show that $\mathrm{Pol}^{\lambda_1}[p]$ is Lorentzian. To

ease notation, we consider
$p \in \mathbb{R}^{(\lambda_1, \lambda_2, \ldots, \lambda_n)}[y, x_2, \ldots, x_n]$
and apply Pol^{λ_1} w.r.t.
the variable y. By induction
and deriv. commuting, the Char. thm.
implies we only need to prove:
(1) $Pol^{\lambda_1}[p]$ is indecomposable
$(deg(p) \geq 2)$
(2) $Pol^{\lambda_1}[p]$ has Hessian with
at most one pos. e-val,
when $deg(p) = 2$.
(Recall Char. theorem on board)

(1) is clear → Suppose
$Pol^{\lambda_1}[p] = g + h$, dependent
on disjoint sets of vars.
Since $Pol^{d}[p]$ is symmetric in
the y vars, it must be that
one of g, h is dep. on all
y vars. But then this

contradicts the fact that p is indecomposable by assumption.

For (2?), the following lemma will immediately imply the result.

<u>Lemma</u>: If $\deg(p) = 2$, then p is Lorentzian iff p is real stable.

<u>Pf.</u>: ($\Rightarrow$) Note that $p(a \cdot t + b \cdot s)$
$$= C_0 s^2 + C_1 st + C_2 t^2$$
satisfies Newton's inequalities, $\forall a, b \in \mathbb{R}^n_{>0}$. Thus $\left(\frac{C_1}{2}\right)^2 \geq C_0 C_2$ $\Leftrightarrow C_1^2 \geq 4 C_0 C_2 \Rightarrow p(a \cdot t + b)$ is real-rooted. Now, for any $b \in \mathbb{R}^n$, $a \in \mathbb{R}^n_{>0}$, $p(a \cdot t + b)$ is real-rooted iff $p(a(t+c)+b)$ $= p(a \cdot t + (b + a \cdot c))$ is real-rooted

for any large $c>0$. Thus $p(a\cdot t+b)$ is real-rooted $\forall a\in\mathbb{R}^n_{>0}$ and $b\in\mathbb{R}^n$, and this completes the proof.

(See HW for other direction.) $\square$

(Linear alg proof?)

Thus Pol^λ preserves Lorentzian. $\square$

<u>Recall</u>: For multiaffine p and linear operator T on multiaffine polynomials, we have

$$T[p](x) = \prod_{i=1}^{n} \left(\partial_{y_i} + \partial_{z_i}\right)\Big|_{z=0} \left[\mathrm{Symb}^1[T](x,z)\cdot p(y)\right]$$

<u>Theorem</u>: If $\mathrm{Symb}^1[T]$ is Lorentzian, then T preserves Lorentzian.

(No converse!)

<u>Pf.</u>: Same argument as real stable case. $\square$

Theorem: If $Symb^\lambda[T]$ is Lorentzian, then T preserves Lorentzian. (No converse.)

Pf: Recall

$$Symb^1[T \circ Diag^\lambda](x,z) = Pol_z^\lambda(Symb^\lambda[T](x,z))$$

Since Pol_z^λ preserves Lorentzian, $T \circ Diag^\lambda$ preserves Lorentzian by the prev. thm.. Thus $T = T \circ Diag^\lambda \circ Pol^\lambda$ preserves Lorentzian as well. $\square$