General Course outline:

- Theory/application of log-concave polynomials (5-6 wks) ↑ maybe more
  - real stable polynomials
  - Lorentzian polynomials
  - more....
- Theory/application of polynomial capacity (5-6 weeks)
  - bounds/approx.
  - applications to various combinatorial problems
  - connection to entropy.

Grading: Assignments; using resources/colleagues highly encouraged,
- only requirement is learning, write answers in your own words

The big idea: "The Geometry of Polynomials"

Algebraic
↳ roots/zeros of polynomial

+ convexity

Combinatorial ⟷ **Analytic**

↳ coefficients of polynomial

↳ evaluations of polynomial

Idea: Use the interplay of these three features of polynomials (to prove facts about objects which have nothing to do with polyns.)

Typical Method:

1) Encode object as polynomial with nice properties
2) Apply operations which preserve those properties
3) Extract info relating back to object

This week: Univariate and real-rooted polynomials. (+ convexity)

The fundamental theorem of algebra:

For all $p \in \mathbb{C}[t]$ with $\deg(p)=d$, $\exists r_1,\dots,r_d \in \mathbb{C}$, $c \neq 0$, s.t.

$$p(x) = \sum_{k=0}^{d} p_k t^k = c \cdot \prod_{i=1}^{d} (t - r_i).$$

notation

(roots of p)

Let $\mathbb{C}^d[t]$ denote the vector space of polynomials of degree at most d.

If $p \in \mathbb{C}^d[t]$ and $\deg(p)=k$, then we say that p has $d-k$ roots at infinity.

Corollary: For all $0 \neq p \in \mathbb{C}^d[t]$, p has exactly d roots counting multiplicity and roots at infinity.

<u>"Converse"</u>: Given $r_1, \dots, r_d \in \mathbb{C}$ $\exists p \in \mathbb{C}^d[t]$ monic with roots $r_1, \dots, r_d$, given by

$$p(t) = \prod_{i=1}^{d} (t - r_i)$$

$$= \sum_{k=0}^{d} (-1)^{d-k} e_{d-k}(r_1, \dots, r_d)\, t^k,$$

where $e_k(x_1, \dots, x_n) = \sum_{\substack{S \subseteq [n] \\ |S| = k}} x^S$ (def.)

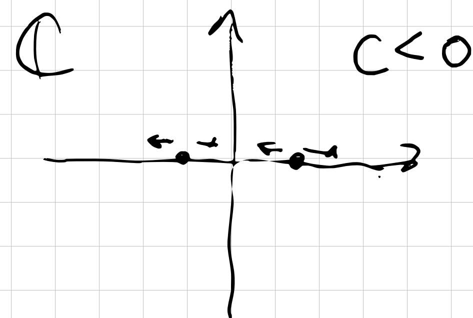
is the <u>elementary symmetric polynomial</u>.

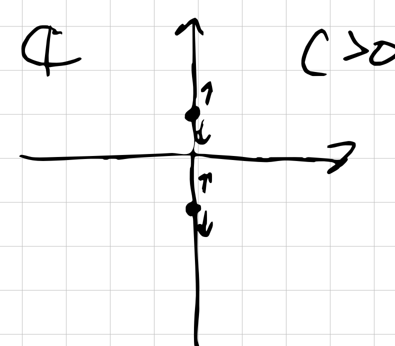
<u>Important</u>: Formula for roots $\to$ coeff., but no formula for coeff $\to$ roots.

(However:) <u>Fact</u>: the roots are continuous as functions of the coefficients. (What does this mean?)

<u>Note</u>: Consider roots in <u>compact</u> space $\mathbb{C} \cup \{\infty\}$.

E.g.: $t^2 + c$, $c \in \mathbb{R}$





(Recall from complex analysis:)

Hurwitz's theorem: The limit of a sequence of polynomials with no roots in open $U \subseteq \mathbb{C}$ is either identically zero or has no roots in U.

Real-rooted polynomials

A polynomial $p \in \mathbb{R}[t]$ is *real-rooted* if all of its roots are real (including ∞). (Sometimes we consider $p \equiv 0$ to be real-rooted.)

Lemma: If $p \in \mathbb{R}[t]$, then non-real roots come in conjugate pairs.

Proof: Let $\alpha \notin \mathbb{R} \cup \{\infty\}$ be s.t. $p(\alpha) = 0$. Letting $\bar{\alpha}$ be the complex conjugate, we have:

$$\overline{p(\bar{\alpha})} = \overline{\sum_{k=0}^{d} p_k (\bar{\alpha})^k} = \sum_{k=0}^{d} p_k \alpha^k = p(\alpha) = 0.$$

Thus $p(\bar{\alpha}) = 0$. $\square$

Proposition: The set of real-rooted polynomials in $\mathbb{R}^d[t]$ is equal to the closure of its interior, the subset having simple roots.

Proof sketch: If $p \in \mathbb{R}^d[t]$ has simple roots, then real perturbations cannot move roots off the real line, by the lemma.

Thus all simple-rooted polynomials are in the interior of real-rooted polynomials in $\mathbb{R}^d[t]$. By continuity of roots, the closure of this set equals the set of all real-rooted polynomials.

Finally, if some real-rooted polynomial has a multiple root, some pair can be perturbed into conjugate pair. $\square$

Convexity properties:

Lemma: If $p \in \mathbb{R}[t]$ is real-rooted, then $\partial_t p$ is real-rooted.

(↳ derivative w.r.t. t)

Proof.: Let $\deg(p) = d$. For any pair of consecutive roots, $r_i < r_{i+1}$, Rolle's theorem implies $\partial_t p$ has a root r such that $r_i < r < r_{i+1}$.

And for any m-multiple root r of p, $\partial_t p$ has a multiple root at

r of multiplicity at least $m-1$, by product rule:

$$\partial_t\left[c\cdot(t-r)^m\prod_{i=1}^{d-m}(t-r_i)\right]$$

$$=c\,m(t-r)^{m-1}\prod_{i=1}^{d-m}(t-r_i)+c(t-r)^m\partial_t\prod_{i=1}^{d-m}(t-r_i)$$

$$=c(t-r)^{m-1}[\cdots].$$

Thus if p has k distinct roots with multiplicities $m_1,\dots,m_k$, then $\partial_t p$ has at least

$$(k-1)+\sum_{i=1}^{k}(m_i-1)=(k-1)+(d-k)=d-1$$

real roots. Therefore p is real-rooted. $\square$

<u>Corollary</u>: If p real-rooted, then the roots of $\partial_t p$ lie in the convex hull of the roots of p. (Extends to complex polynomials)

Modern language: The linear operator ∂_t preserves real-rootedness.

Some other preservers:

- shifted derivative: $p + a\partial_t p$, $a \in \mathbb{R}$
- scaling: $p(at)$ for $a \in \mathbb{R}$
- inversion: $t^d p(t^{-1})$, $p \in \mathbb{R}^d[t]$. $= \sum_{k=0}^{d} p_k t^{d-k}$

(• $SL_2(\mathbb{R})$ action)

Key fact for real-rooted polynomials:

Theorem (Newton's inequalities):

If $p \in \mathbb{R}[t]$ is real-rooted and $\deg(p) = d$, then

$$\left(\frac{p_k}{\binom{d}{k}}\right)^2 \geq \left(\frac{p_{k-1}}{\binom{d}{k-1}}\right)\left(\frac{p_{k+1}}{\binom{d}{k+1}}\right)$$

for all $k = 1, 2, \dots, d-1$.

Definition: The sequence $\frac{p_k}{\binom{d}{k}}$ is called log-concave, the sequence p_k is called ultra log-concave.

(This is one of the main reasons geometry of polynomials has combinatorics applications. Algebraic properties (real-rooted) implies combinatorial statements (log-concavity).)

Proof: Fix $k \in \{1, 2, \dots, d-1\}$.

Let $p = \sum_{j=0}^{d} \binom{d}{j} c_j t^j \in \mathbb{R}^d[t]$.

Let $f = \partial_t^{k-1} p = \sum_{j=k-1}^{d} \frac{d!}{(j-k+1)!(d-j)!} c_j t^{j-k+1}$.

Let $g = t^{d-k+1} f(t^{-1}) =$
$$= \sum_{j=k-1}^{d} \frac{d!}{(j-k+1)!(d-j)!} c_j t^{d-j}.$$

Let $q = \partial_t^{d-k-1} g$
$$= \sum_{j=k-1}^{k+1} \frac{d!}{(j-k+1)!(k-j+1)!} c_j t^{k-j+1}.$$

All preserve real-rootedness
$\Rightarrow q$ is real-rooted.

Thus,

$$\frac{2!\cdot q(t)}{d!} = \frac{2!}{0!\cdot 2!} c_{k-1} t^2 + \frac{2!}{1!\cdot 1!} c_k t + \frac{2!}{2!\cdot 0!} c_{k+1}$$

$$= c_{k-1} t^2 + 2\cdot c_k t + c_{k+1}$$

is real-rooted.

Thus $b^2 - 4ac \geq 0 \rightarrow$

$$(2\cdot c_k)^2 - 4 c_{k-1} c_{k+1} \geq 0$$

$$\rightarrow \quad c_k^2 \geq c_{k-1} c_{k+1}. \qquad \square$$

(What about the converse? Not true in general. However, if you homogenize $p \in \mathbb{R}_{\geq 0}[t]$, then p having ultra log-concave coeff. (w/ no internal zeros) will be equivalent to p being Lorentzian. This is one motivation of the definition of Lorentzian.)

Another instance of convexity:

Theorem (Gauss–Lucas): Given $p \in \mathbb{C}[t]$, the roots of $\partial_t p$ are contained in the convex hull of the roots of p in $\mathbb{C}$.

(Recall: extension of $\partial_t p$ real-rooted from Rolle's theorem.)

Proof: Let $p(t) = C \cdot \prod_{i=1}^{d} (t - r_i)$.

Fix r s.t. $\partial_t p(r) = 0$.

If $p(r) = 0$, we're done. Otherwise, $p(r) \neq 0 \Rightarrow$

$$0 = \frac{\partial_t p}{p}(r) = \frac{d}{dt}\left[\log p\right]\Big|_{t=r} = \sum_{i=1}^{d} \frac{1}{r - r_i}$$

$$\Rightarrow 0 = \sum_{i=1}^{d} \frac{1}{\overline{(r - r_i)}} = \sum_{i=1}^{d} \frac{r - r_i}{|r - r_i|^2}$$

$$\Rightarrow \quad r \cdot \sum_{i=1}^{d} \frac{1}{|r - r_i|^2} = \sum_{i=1}^{d} \frac{r_i}{|r - r_i|^2}$$

Lecture 2 – Grace's Theorem

Theorem (Gauss–Lucas): Given $p \in \mathbb{C}[t]$, the roots of $\partial_t p$ are contained in the convex hull of the roots of p in $\mathbb{C}$.

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Proof: Let $p(t) = C \cdot \prod_{i=1}^{d} (t - r_i)$.

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$$\Rightarrow 0 = \sum_{i=1}^{d} \frac{1}{\overline{(r - r_i)}} = \sum_{i=1}^{d} \frac{r - r_i}{|r - r_i|^2}$$

$$\Rightarrow \quad r \cdot \sum_{i=1}^{d} \frac{1}{|r - r_i|^2} = \sum_{i=1}^{d} \frac{r_i}{|r - r_i|^2}$$

$$\Rightarrow r = \sum_{i=1}^{d} r_i \cdot \left(\frac{\frac{1}{|r-r_i|^2}}{\sum_{j=1}^{d} \frac{1}{|r-r_j|^2}} \right)$$

That is, r is a convex combination of the r_i's. $\square$

((Conceptually: the roots of the derivative can be considered as the equillibria of some kind of "potential" determined by point charges at the roots of p.)

<u>Corollary</u>: The linear operator ∂_t preserves real-rooted polynomials, half-plane-rooted polynomials, disc-rooted polynomials.

Generalization of Gauss-Lucas: Laguerre's theorem

Consider $\partial_t : \mathbb{C}^d[t] \to \mathbb{C}^{d-1}[t]$
and $\tilde{\cdot} : \mathbb{C}^d[t] \to \mathbb{C}^d[t]$
via $\tilde{p}(t) = t^d p(1/t)$

Now define:
$\tilde{\partial}_t : \mathbb{C}^d[t] \to \mathbb{C}^{d-1}[t]$ via

$$\tilde{\partial}_t p = \widetilde{\left(\partial_t \tilde{p}\right)}$$

"Derivative with respect to 0."
(Usual derivative is "with respect to ∞")

<u>Q</u>: Why this name?

$SL_2(\mathbb{C})$ acts on the Riemann sphere via Möbius transformations:

$$\phi \cdot r = \frac{\alpha r + \beta}{\gamma r + \delta} \text{ where } \phi = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

(group action) $\qquad r \in \mathbb{C} \cup \{\infty\}$.

Action extends naturally to $\mathbb{C}^d[t]$:

$$(\phi^{-1}\cdot p)(t) = (\gamma t+\delta)^d \cdot p(\phi\cdot t)$$
$$= \sum_{k=0}^{d} p_k (\alpha t+\beta)^k (\gamma t+\delta)^{d-k}.$$

<u>Fact</u>: r is a root of p iff $\phi\cdot r$ is a root of $\phi\cdot p$.

<u>Pf. Sketch</u>: $(\phi\cdot p)(t) = (\cdots)^d p(\phi^{-1}\cdot t)$

So, $(\phi\cdot p)(\phi\cdot r) = (\cdots)^d p(r) = 0$.

(Details of group actions, roots at ∞, etc. in assignment)

<u>Definition</u>: A <u>circular region</u> is an open/closed disc, half-plane, or complement of a disc in $\mathbb{C}\cup\{\infty\}$.

<u>Lemma (Möbius transf. from Complex analysis)</u>: If C is a circular region, and $\phi\in SL_2(\mathbb{C})$, then $\phi\cdot C$ is a circular region.

(almost to Laguerre's theorem)

Lemma: Fix $\phi = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL_2(\mathbb{C})$, and define $\begin{bmatrix} a \\ c \end{bmatrix} = \phi^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then
($[\begin{smallmatrix} a \\ c \end{smallmatrix}] = [\begin{smallmatrix} \delta \\ -\gamma \end{smallmatrix}]$)

$$(\phi^{-1} \circ \partial_t \circ \phi) \cdot p = (a\partial_t + c\tilde{\partial}_t) p.$$

Proof: Assignment. (simple chain rule)

Call this the polar derivative with respect to the pole a/c.
(Essentially: ϕ moves a/c to ∞, then take derivative, then move ∞ back to a/c.
Can also think in terms of rotations of the Riemann sphere.)

Theorem (Laguerre's theorem):
Fix $\frac{a}{c} \in \mathbb{C} \cup \{\infty\}$, and suppose $0 \not\equiv p \in \mathbb{C}^d[t]$ $d \geq 1$
has all roots in a circular region C with $\frac{a}{c} \notin C$. Then

$$(a\partial_x + c\tilde{\partial}_x)p \not\equiv 0$$

also has all its roots in C.

<u>Proof</u>: Let $\phi \in SL_2(\mathbb{C})$ be s.t.

$$\begin{bmatrix} a \\ c \end{bmatrix} = \phi^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \phi\left(\tfrac{a}{c}\right) = \infty.$$

Thus $\phi \cdot p$ has all roots in a circular region $C' = \phi \cdot C$ not containing ∞. Thus $\deg(p) = d$ and C' is a disc or a half-plane. By Gauss-Lucas, $\partial_x(\phi \cdot p) \not\equiv 0$ has all roots in C'. Thus,

$$\phi^{-1}(\partial_x(\phi \cdot p)) = (\phi^{-1} \circ \partial_x \circ \phi)p \not\equiv 0$$

has all roots in C. $\square$

<u>Corollary</u>: Real polar derivatives preserve real-rootedness.

(Question: Now what? A: The power of Laguerre's theorem comes in the fact that the polar derivative can be expressed as a linear combination of nice derivative operators.)

Fix $p, q \in \mathbb{C}_d^d[t]$ with

$$p(t) = \prod_{i=1}^{d} (c_i t - a_i). \quad \left(\text{roots are } a_i/c_i\right)$$

Define:

$$\langle p, q \rangle = \left[\sum_{k=0}^{d} p_k \tilde{\partial}_t^k (-\partial_t)^{d-k}\right] q(t)$$

$$= \prod_{i=1}^{d} (a_i \partial_t + c_i \tilde{\partial}_t) q(t) \in \mathbb{C}.$$

$\hookrightarrow$ Bilinear form on $\mathbb{C}^d[t]$.

Theorem (Grace's theorem, ~1900):

Fix $0 \not\equiv p, q \in \mathbb{C}^d[t]$ and a circular region C s.t. p has no roots in C and q has all roots in C. Then

$$\langle p, q \rangle \neq 0.$$

Proof: Let $p(t) = \prod_{i=1}^{d} (c_i t - a_i)$ as above. Thus

$$\langle p, q \rangle = \prod_{i=1}^{d} (a_i \partial_t + c_i \tilde{\partial}_t)\, q(t).$$

By Laguerre's theorem applied inductively, we have that

$$\prod_{i=1}^{k} (a_i \partial_t + c_i \tilde{\partial}_t)\, q(t) \quad \text{is}$$

not identically 0 and has no roots in C, for all k.

Thus $\langle p, q \rangle \neq 0$.

Fact: $\langle p, q\rangle = d! \sum_{k=0}^{d} \binom{d}{k}^{-1} (-1)^{d-k} p_k q_{d-k}$

Proof: $\langle t^k, t^j \rangle = \partial_t^k (-\partial_t)^{d-k} t^j$

$= (-1)^{d-k} (d-k)!\, k!\, \delta_{j,d-k}.$

(thus $\langle \cdot, \cdot \rangle$ is symmetric up to $(-1)^d$)

(Grace's theorem is the underlying result from which many results are proven for real-rooted polynomials and generalizations. This includes a characterization of linear operators preserving real-rootedness. But, how is this possible?)

Important Corollary: Walsh coincidence theorem.

(The beginning of the multivariate theory)

Def.: Given a polynomial $p \in \mathbb{C}^d[t]$, its polarization is the unique polynomial P s.t.

- $P \in \mathbb{C}^{(1,1,\dots,1)}[x_1, x_2, \dots, x_d]$
  ↳ (deg. at most 1 in x_i, $\forall i$)
  ↳ also called multiaffine.
- P is symmetric
- $p(x) = P(x, x, \dots, x)$.

Lemma: If $p(t) = C \cdot \prod_{i=1}^{d} (c_i t - a_i) \in \mathbb{C}^d[t]$, then

(1) $P(x_1, \dots, x_d) = \frac{1}{d!} \sum_{\sigma \in S_d} C \cdot \prod_{i=1}^{d} (c_i x_{\sigma(i)} - a_i)$

(2) $P(x_1, \dots, x_d) = \frac{1}{d!} \prod_{i=1}^{d} (x_i \partial_t + \tilde{\partial}_t)\, p(t)$

Proof: (1) is obvious. For (2), just need to show that
$p(x) = P(x, x, \dots, x)$.

$$P(x,x,\dots,x) = \frac{1}{d!}\prod_{i=1}^{d}(x\partial_t + \tilde{\partial}_t)\,p(t)$$

$$= \frac{1}{d!}\sum_{k=0}^{d}\binom{d}{k}x^k\left(\partial_t^k\tilde{\partial}_t^{d-k}p(t)\right)$$

$$= \frac{1}{d!}\sum_{k=0}^{d}\binom{d}{k}x^k\cdot k!(d-k)!\,p_k$$

$$= \sum_{k=0}^{d}p_k x^k = p(x). \qquad \square$$

(Reminiscent of Grace's theorem $\longrightarrow$ this is no accident.)

<u>Theorem (Walsh Coincidence theorem):</u>

If $p \in \mathbb{C}^d[t]$ has no roots in a circular region C, then $P(x_1, x_2, \dots, x_d) \neq 0$ when $x_1, x_2, \dots, x_d \in C$. ($P$ is <u>C^d-stable</u>.)

Lecture 3: Real stable polynomials

(Multivariate version of real-rooted)

(To motivate multivariate polynomials, Let's finish last week.)

__Recall: Grace's theorem:__

Let $0 \neq p, q \in \mathbb{C}^d[t]$ and circular region C be s.t.

p has no roots in C and q has no roots in C^c (also a circular region). Then

$$0 \neq \langle p, q \rangle = c_0 \cdot \prod_{i=1}^{d} \left(a_i \partial_t + c_i \tilde{\partial}_t\right) q(t).$$

where $p(t) = c_0 \prod_{i=1}^{d} (c_i t - a_i)$

(statement is symmetric in p, q)

__Recall: Polarization:__ Given $p \in \mathbb{C}^d[t]$, the *polarization* $\mathrm{Pol}^d[p] \in \mathbb{C}^{(1,1,\dots,1)}[x_1, \dots, x_d]$ is the unique symmetric multiaffine polynomial s.t. $\mathrm{Pol}^d[p](x, x, \dots, x) = p(x)$.

(Equivalent to Grace's theorem)

Theorem (Walsh Coincidence theorem):

If $p \in \mathbb{C}^d[t]$ has no roots in a circular region C, then $\mathrm{Pol}^d[p](x_1, x_2, \dots, x_d) \neq 0$ when $x_1, x_2, \dots, x_d \in C$. ($p$ is $\underline{C^d\text{-stable}}$.)

Proof: By Lemma last time,

$$\mathrm{Pol}^d[p](x_1, \dots, x_d) = \frac{1}{d!} \prod_{i=1}^{d} (x_i \partial_t + \tilde{\partial}_t)\, p(t)$$

$$= \frac{1}{d!} \langle q, p \rangle,$$

where $q \in \mathbb{C}^d[t]$ has roots $x_1, x_2, \dots, x_d$. If p has no roots in C and $x_1, \dots, x_d \in C$, then q has no roots in C^c, and thus $\langle q, p \rangle \neq 0$ by Grace's theorem. $\square$

Fact: $p \in \mathbb{R}^d[x]$ is real-rooted iff it is H_+-stable, since complex roots come in conjugate pairs.
(upper half-plane)

(Studying real-rooted polynomials is equivalent to studying real stable multivariate polynomials of deg. at most 1 in each variable. Much more structure available.)

(Where we're headed: the power of Grace's theorem/ the Walsh coincidence theorem)

Recall the "geometry of polynomials" method:

1) Encode objects in polynomials with nice properties

$\boxed{\text{2) Apply operators which preserve properties}}$

3) Extract info about the original object.

Q: What operators preserve nice properties? E.g., real-rootedness?

Foreshadowing: Symbol of a linear operator

Let $L(\mathbb{C}^d[t], \mathbb{C}^d[t])$ denote space of lin. ops..

$$L(\mathbb{C}^d[t], \mathbb{C}^d[t]) \cong \mathbb{C}^d[t] \otimes (\mathbb{C}^d[t])^*$$

$$\cong \mathbb{C}^d[t] \otimes \mathbb{C}^d[t] \cong \mathbb{C}^{(d,d)}[s,t]$$

non-canonical, requires choice of dual basis $\Longleftrightarrow$ non-degenerate bilinear form.

Lemma: $\langle \cdot, \cdot \rangle$ is non-degenerate.

Pf.: $\forall\, 0 \not\equiv p \in \mathbb{C}^d[t]$, let C contain all roots of p, and pick q which has no roots in C. By Grace's theorem, $\langle p, q \rangle \neq 0$. $\square$

(Note: $\langle p, (t-\alpha)^d \rangle = c_0 \cdot p(\alpha)$ for some $c_0 \neq 0$.)

By choosing this bilinear form, we obtain the following: Given $T \in L(\mathbb{C}^d[t], \mathbb{C}^d[t])$, let $f_T \in \mathbb{C}^{(d,d)}[s,t]$ be the corresponding polynomial. Then:

$$T[p](s) = \langle f_T(s,t), p(t) \rangle$$

(possibly up to some non-zero constant) where $\langle \cdot, \cdot \rangle$ acts on the t variable.

<u>Now</u>: If f_T has no zeros in $C \times C^c$ (C some circular region), then $\forall s_0 \in C$, $f_T(s_0, t) \in \mathbb{C}^d[t]$ has no roots in C^c. If p has no roots in C, then $T[p](s_0) \neq 0$ by Grace's theorem. That is, T preserves C-stability.

(That is, stability properties of a single polynomial f_T gives stability preservation properties of T.)

<u>E.g.</u> Let $T \in L(\mathbb{R}^d[t], \mathbb{R}^d[t])$, and let f_T be $(H_+ \times H_+^c)$-stable. If $p \in \mathbb{R}^d[t]$ real-rooted, then it is H_+-stable. Thus

$$T[p](s) \cong \langle f_T(s,t), p(t) \rangle$$

implies $T[p]$ is H_+-stable. Since $T[p] \in \mathbb{R}^d[t]$ and complex roots come in conjugate pairs, $T[p]$ must be real-rooted.

<u>Upshot</u>: Grace's theorem and bilinear form mentality quickly give real-rootedness preservers results $\Rightarrow$ ultra log-concavity.

(Slight technical issue with this result: doesn't allow for the zero polynomial. E.g., doesn't capture ∂_x. One way to handle this: topological arguments. We will go a different route, via equiv. Walsh coincidence theorem)

Stability properties of bivariate polynomial $\Rightarrow$ Stability preservation results.

↳ Motivates <u>multivariate theory</u>.

<u>Real stable polynomials</u>

$\mathbb{C}^{\lambda}[X] = \mathbb{C}^{(\lambda_1, \dots, \lambda_n)}[x_1, \dots, x_n]$

$\mathbb{R}^{\lambda}[X] = \mathbb{R}^{(\lambda_1, \dots, \lambda_n)}[x_1, \dots, x_n]$

↳ deg. at most λ_i in x_i

Definition: A polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ is said to be stable if

$$x_1, \dots, x_n \in H_+ \implies p(x_1, \dots, x_n) \neq 0.$$

↳upper half-plane

If $p \in \mathbb{R}[x_1, \dots, x_n]$, then we say p is real stable.

Note: $p \equiv 0$ is usually considered stable and real stable.

Fact: $p \in \mathbb{R}^d[t]$ real-rooted iff real stable.

E.g. (next week) Matching polynomial, spanning tree gen. polynomial, $\det(\sum_i x_i A_i)$ for positive semi-definite A_i, more....

<u>Corollary of Coincidence theorem</u>: Given $p \in \mathbb{R}^d[t]$, p is real-rooted if and only if $\mathrm{Pol}^d[p]$ is real stable.

(<u>Proof</u>: ($\Rightarrow$) p real-rooted $\to$ p $\mathcal{H}_+$-stable $\to$ $\mathrm{Pol}^d[p]$ is stable $\to$ $\mathrm{Pol}^d[p]$ real stable

($\Leftarrow$) $\mathrm{Pol}^d[p]$ real stable $\to$ p real stable $\to$ p is real-rooted.)

<u>Proposition</u>: A polynomial $p \in \mathbb{R}^\lambda[x]$ is real stable if and only if $\forall a \in \mathbb{R}^n_{>0}, b \in \mathbb{R}^n$, the polynomial

$$q(t) = p(a_1 t + b_1, \dots, a_n t + b_n) \in \mathbb{R}^{\lambda_1 + \dots + \lambda_n}[t]$$

is real-rooted (or $\equiv 0$).

(<u>Upshot</u>: real-rootedness "everywhere" $\to$ ultra log-concavity)

<u>Proof</u>:

($\Rightarrow$) For $z \in H_+$, we have
$$q(z) = p(a_1 z + b_1, \dots, a_n z + b_n).$$
Since $a_k > 0$ and $b_k \in \mathbb{R}$, $a_k z + b_k \in H_+$ for all k.
Thus $q(z) \neq 0 \Rightarrow q$ is real-rooted, since complex roots come in conj. pairs.

($\Leftarrow$) For all $z_1, \dots, z_n \in H_+$, we have $z_k = a_k i + b_k$, $a_k > 0$, $b_k \in \mathbb{R}$.
Thus $0 \neq q(i) = p(a_1 i + b_1, \dots, a_n i + b_n)$
$$= p(z_1, \dots, z_n)$$
$\Rightarrow p$ is real stable. $\square$

<u>Geom. of polys. method</u>:

1) Encode object as real stable polynomial.
2) Apply real stability preservers.
3) (E.g.) Restrict to line to obtain ultra log-concave sequence.

Some (linear) preservers of real stable polynomials:

- <u>evaluation</u>: $p \mapsto p(b, x_2, \ldots, x_n)$ for $b \in \mathbb{R}$
- <u>positive affine translation</u>:
  $$p \mapsto p(ax_1+b, x_2, \ldots, x_n)$$
  $a>0, b \in \mathbb{R}$
- <u>permutation</u>: $p \mapsto p(x_2, x_1, x_3, \ldots, x_n)$
- <u>full inversion</u>: $p \mapsto x^\lambda p(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n})$
- <u>diagonalization</u>: $p \mapsto p(x_1, x_1, x_3, \ldots, x_n)$
- <u>variable expansion</u>:
  $$p \mapsto p(x_{11}+x_{12}+\cdots+x_{1m}, x_2, \ldots, x_n)$$
- <u>positive directional derivatives</u>
  $$p \mapsto D_v p = \sum_{k=1}^n v_k \partial_{x_i} p,$$
  where $v_k \geq 0 \ \forall k$.

<u>Proof</u>: Positive affine translation easy. For evaluation,

limit $a \to 0^+$ and apply Hurwitz's theorem.

(Permutation/diagonalization obvious.)

For full inversion, note that

p real stable iff

$p(a \cdot t + b)$ real-rooted $\forall a > 0, b \in \mathbb{R}^n$

iff $p(a(-t)+b)$ real-rooted

iff $p(-x_1, -x_2, \ldots, -x_n)$ real stable.

Thus $p \in \mathbb{R}[x_1, \ldots, x_n]$ is stable

iff p is H_-^n-stable

$\hookrightarrow$ lower half-plane.

Since $z \to \frac{1}{z}$ maps the upper half-plane to the lower half-plane,

$z^n p\left(\frac{1}{z_1}, \ldots, \frac{1}{z_n}\right)$ is real stable.

For variable expansion,

$x_{11}, x_{12}, \ldots, x_{1m} \in H_+ \to \sum_k x_{1k} \in H_+$.

Lecture 4: The Borcea–Brändén (BB) characterization

Recall: A polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ is **stable** if p is H_+^n-stable, and if $p \in \mathbb{R}[x_1, \dots, x_n]$ then p is **real stable**.

Basic preservers of (real) stability:

If p is (real) stable, then so are:

- $p(b, x_2, \dots, x_n)$ $\quad b \in \mathbb{R}$
- $p(ax_1 + b, x_2, \dots, x_n)$ $\quad a > 0, b \in \mathbb{R}$
- $p(x_2, x_1, x_3, \dots, x_n)$
- $x^\lambda p(x_1^{-1}, \dots, x_n^{-1})$ $\quad p \in \mathbb{R}^\lambda[x_1, \dots, x_n]$
- $p(x_1, x_1, x_3, \dots, x_n)$
- $p(x_{11} + x_{12} + \dots + x_{1m}, x_2, \dots, x_n)$
- $p \cdot q$, where q is (real) stable.
- $D_v p = \sum_{k=1}^{n} v_k \partial_{x_k} p$, $\quad v_k \geq 0$.

Proof of last property:

Consider

$p(x_1 + v_1 y, x_2 + v_2 y, \dots, x_n + v_n y) \in \mathbb{C}[x_1, \dots, x_n, y]$

which is (real) stable by above properties. Now, $D_v p = \partial_y p|_{y=0}$, so we may restrict to the case where $D_v = \partial_{x_1}$.

For all $x_2, \dots, x_n \in H_+$,

$q(t) = p(t, x_2, \dots, x_n) \in \mathbb{C}^d[t]$ is stable.

Thus by Gauss-Lucas, $\partial_t q$ is stable. Therefore, $\partial_{x_1} p$ is stable. □

(We have shown certain basic operations preserve (real) stability.)

<u>Q</u>: What about more interesting (real) stability preservers?

The symbol of an operator

Given a linear operator $T \in L(\mathbb{C}^\lambda[x], \mathbb{C}^{\lambda'}[x])$, define the <u>Borcea-Brändén (BB) symbol</u> of T to be:

$$\mathrm{Symb}^\lambda[T](x,z) := T\left[\prod_{i=1}^{n}(x_i+z_i)^{\lambda_i}\right]$$

(linear)

$$= \sum_{0\le\mu\le\lambda} \binom{\lambda}{\mu} z^{\lambda-\mu} T[x^\mu],$$

where T acts on x variables, $\mu \le \lambda$ is entrywise, and

$$\binom{\lambda}{\mu} := \prod_{i=1}^{n} \binom{\lambda_i}{\mu_i}.$$

Theorem (Borcea-Brändén '09):

Given $T \in L(\mathbb{C}^\lambda[x], \mathbb{C}^{\lambda'}[x])$, T preserves stability (allowing $\equiv 0$) if and only if one of the following holds:

1) $\mathrm{Symb}^\lambda[T](x,z)$ is stable
2) The image of T is a one-dimensional space of stable polynomials.

Roughly: T preserves stability "iff" its symbol is stable.

(Reminiscent of the argument from last lecture, which showed that T preserves $\mathbb{C}$-stability iff f_T has some stability properties.)

(Today's main goal: prove this theorem and a similar theorem about real stable polynomials.)

Important direction, which we will prove:

Symbol stable $\Rightarrow$ T preserves stability

"Simpler" case first: multiaffine polynomials, $\lambda = (1, 1, \dots, 1)$.

Proof (multiaffine): (λ' anything)

For any $p \in \mathbb{C}^{(1,\dots,1)}[y]$, we have

$T[p](x) =$

$$\prod_{i=1}^{n} (\partial_{z_i} + \partial_{y_i})\big|_{z=y=0} \left[\mathrm{Symb}^1[T](x,z) \cdot p(y) \right]$$

(Idea: this diff op. gives a choice of non-degenerate bilinear form, giving rise to $\mathrm{Symb}^1[T]$ like in last lecture.)

We prove this on the basis of monomials.

Let $p(y) = y^\alpha$, which implies

$$\prod_{i=1}^{n} (\partial_{z_i} + \partial_{y_i})\big|_{z=y=0} \left[\mathrm{Symb}^1[T](x,z) \cdot y^\alpha \right]$$

$$= \partial_z^{1-\alpha}\big|_{z=0} \mathrm{Symb}^1[T](x,z)$$

$$= \partial_z^{1-\alpha}\big|_{z=0} \sum_{0 \le \mu \le 1} \binom{1}{\mu} z^{1-\mu} \cdot T[x^\mu]$$

$$= T[x^\alpha]$$

$$= T[p](x).$$

That is, for $G := \prod_{i=1}^{n} (\partial_{z_i} + \partial_{y_i})\big|_{z=y=0}$,

we have

$$T[p] = G[\mathrm{Symb}^1[T](x,z) \cdot p(y)].$$

Note G is a (real) stability preserver. Thus if p and $\mathrm{Symb}^\lambda[T]$ are stable, then $T[p]$ is stable.

That is, T preserves stability.

<u>Next</u>: Case of general λ.

Given $p \in \mathbb{C}^\lambda[x]$, define

$$\mathrm{Pol}^\lambda[p] = (\mathrm{Pol}^{\lambda_1} \circ \mathrm{Pol}^{\lambda_2} \circ \cdots \circ \mathrm{Pol}^{\lambda_n})[p],$$

where Pol^{λ_i} acts on x_i variable.

Thus $\mathrm{Pol}^\lambda[p]$ is a multiaffine polynomial in $\lambda_1 + \cdots + \lambda_n$ variables, and Pol^λ preserves (real) stability.

$\underline{Q}$: How does Pol^λ relate to Symb^λ?

$$\mathrm{Symb}^\lambda[T](x,z) = T\left[\prod_{i=1}^{n}(x_i+z_i)^{\lambda_i}\right].$$

Letting Diag^λ be the inverse of Pol^λ (setting variables equal),

$$T\circ\mathrm{Diag}^\lambda \in L\left(\mathbb{C}^{(1,\dots,1)}[(x_{ij})_{1\le i\le n,\,1\le j\le\lambda_i}],\ \mathbb{C}^{\lambda'}[x]\right)$$

$$\mathrm{Symb}^1[T\circ\mathrm{Diag}_x^\lambda](x,z)$$
$$= (T\circ\mathrm{Diag}_x^\lambda)\left[\prod_{i=1}^{n}\prod_{j=1}^{\lambda_i}(x_{ij}+z_{ij})\right]$$
$$= T\left[\prod_{i=1}^{n}\prod_{j=1}^{\lambda_i}(x_i+z_{ij})\right]$$

For fixed i, $\prod_{j=1}^{\lambda_i}(x_i+z_{ij}) = \mathrm{Pol}_z^{\lambda_i}[(x_i+z_i)^{\lambda_i}]$

Thus,

$$\mathrm{Symb}^1[T\circ\mathrm{Diag}^\lambda](x,z)$$
$$= \mathrm{Pol}_z^\lambda\left(T\left[\prod_{i=1}^{n}(x_i+z_i)^{\lambda_i}\right]\right)$$
$$= \mathrm{Pol}_z^\lambda\left(\mathrm{Symb}^\lambda[T](x,z)\right).$$

Therefore $\mathrm{Symb}^\lambda[T](x,z)$ being stable implies $\mathrm{Symb}^1[T\circ \mathrm{Diag}^\lambda](x,z)$ is stable by Walsh coincidence, and this implies $T\circ \mathrm{Diag}^\lambda$ preserves stability by the multiaffine case. Thus $T = T\circ \mathrm{Diag}^\lambda \circ \mathrm{Pol}^\lambda$ preserves stability by Walsh coincidence again.

$\square$

<u>Q</u>: What about real stability preserver?

<u>Theorem (Borcea-Brändén '09)</u>:

Given $T\in L(\mathbb{R}^\lambda[x], \mathbb{R}^\lambda[x])$, T preserves real stability if and only if one of the following holds:

1) $\mathrm{Symb}^\lambda[T](x,z)$ is real stable
2) $\mathrm{Symb}^\lambda[T](-x,z)$ is real stable
3) The image of T is a two-dim. space of real stable polynomials.

__Proof__: Clearly, if $\mathrm{Symb}[T](x,z)$ is real stable, then T preserves real stability by the previous theorem. Recall that a polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ is real stable if and only if $p(-x)$ is real stable (since p is H_+^n-stable iff p is H_-^n-stable). Now consider the operator S, given by $S[p](x) = T[p](-x)$. Thus, if S preserves stability, then so does T. We compute

$$\begin{aligned}
&\mathrm{Symb}^\lambda[S](x,z) \\
&\quad = \sum_{0 \le \mu \le \lambda} \binom{\lambda}{\mu} z^{\lambda-\mu} \cdot T[x^\mu](-x) \\
&\quad = \mathrm{Symb}^\lambda[T](-x,z).
\end{aligned}$$

Thus if $\mathrm{Symb}^\lambda[T](-x,z)$ is stable, then S preserves stability by the prev. thm..

<u>Thus</u>: Completely answers the question of linear operators preserving stability and real stability.

<u>Note</u>: More general theorem for C^n-stability preservers for any circular region C, using Möbius transf..

<u>Sketch of another proof</u>:

1) Prove Grace's theorem for multivariate polynomials
2) Use this bilinear form to construct "universal" symbol
3) Use argument we discussed last time (with some details)

<u>Note</u>: "Transcendental" characterization also exists (no degree bounds):

$$\mathrm{Symb}^{\infty}[T](x,z) = T[e^{-x\cdot z}] = \sum_{0\le\mu}(-1)^{\mu}\frac{z^{\mu}}{\mu!}T[x^{\mu}].$$

Lecture 5: Basic applications

Recall:

Theorem Borcea-Brändén '09):

Given $T \in L(\mathbb{C}^{\lambda}[x], \mathbb{C}^{\lambda'}[x])$, T preserves stability (allowing $\equiv 0$) if and only if one of the following holds:

1) $\mathrm{Symb}^{\lambda}[T](x,z)$ is stable
2) The image of T is a one-dimensional space of stable polynomials.

Recall: $\mathrm{Symb}^{\lambda}[T](x,z) = T\left[\prod_{i=1}^{n} (x_i + z_i)^{\lambda_i}\right]$

Proof: For multiaffine,

($\Leftarrow$) $\quad \prod_{i=1}^{n} (\partial_{x_i} + \partial_{z_i}) \Big|_{z=x=0}$ preserves.

Then for general case use Walsh coincidence theorem.

Recall:

Theorem (Borcea-Brändén '09):

Given $T \in L(\mathbb{R}^\lambda[x], \mathbb{R}^\lambda[x])$, T preserves real stability if and only if one of the following holds:

1) $\mathrm{Symb}^\lambda[T](x,z)$ is real stable
2) $\mathrm{Symb}^\lambda[T](-x,z)$ is real stable
3) The image of T is a ($\leq$) 2-dim. space of real stable polynomials.

Proof: (1) follows from prev. result, ($\Leftarrow$) and (2) follows from

$p \mapsto p(-x)$ preserves real stability

Note: "Transcendental" characterization also exists (no degree bounds):

$$\mathrm{Symb}^\infty[T](x,z) = T[e^{-x \cdot z}] = \sum_{0 \leq \mu} (-1)^\mu \frac{z^\mu}{\mu!} T[x^\mu]$$

should lie in Polya-Schur class

(Def. as limits on compact sets of (real) stable polynomials)

Some sanity checks:

(1) $\mathrm{Pol}^d \in L(\mathbb{C}^d[x], \mathbb{C}^{(1,\dots,1)}[x_1,\dots,x_d])$

$$\mathrm{Symb}^d[\mathrm{Pol}^d](x,z) = \mathrm{Pol}^d[(x+z)^d]$$
$$= \prod_{i=1}^{d} (x_i + z)$$

If $\mathrm{Im}(x_i), \mathrm{Im}(z) > 0$, then this non-zero. Thus Pol^d preserves stability.

(2) $\partial_{x_k} \in L(\mathbb{C}^\lambda[x], \mathbb{C}^\lambda[x])$

$$\mathrm{Symb}^\lambda[\partial_{x_k}](x,z)$$
$$= \partial_{x_k} \prod_{i=1}^{n} (x_i + z_i)^{\lambda_i}$$
$$= \lambda_k (x_i + z_i)^{\lambda_k - 1} \prod_{i \neq k} (x_i + z_i)^{\lambda_k}$$

This is stable and thus ∂_{x_k} preserves stability.

Differential operators preserving stability

Theorem: Given $p \in \mathbb{C}^\lambda[x]$, the operator $p(\partial_x)$ preserves stability ($\mathbb{C}^\lambda$) iff $p(\partial_x)x^\lambda$ is stable.

Proof: Compute symbol:

$$\mathrm{Symb}^\lambda[p(\partial_x)] = p(\partial_x)\prod_{i=1}^n (x_i + z_i)^{\lambda_i}$$

$$= \left[p(\partial_x)x^\lambda\right](x+z)$$

(shift commutes with derivatives)

Since $|_{z=0}$ and $x \mapsto x+z$ both preserve stability, $\mathrm{Symb}^\lambda[p(\partial_x)]$ is stable iff $p(\partial_x)x^\lambda$ is.

What about one-dimensional image?

$\Rightarrow p(\partial_x) = a \cdot \partial_x^\lambda \Rightarrow p(x) = a \cdot x^\lambda$ stable.

(for real stability, requires a bit more)

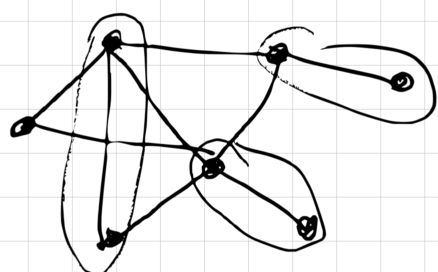
E.g.: $\sum_{S \subseteq \binom{[n]}{k}} \partial^S x^{(d,\dots,d)} = d^S x^{(d,\dots,d)} e_k(x^{-1})$

$\hookrightarrow \mathrm{Pol}^n(x^k)$

Application: the matching polynomial:

Let $G=(V,E)$ be a simple graph, and let μ_k denote the number of matchings in G with k edges.

E.g.:



3-matching

Consider the <u>matching polynomial</u>:

$$\mu_G(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \mu_k (-1)^k t^{n-2k},$$

where $n=|V|$.

Theorem (Heilmann-Lieb '72): For any graph G, $\mu_G(t)$ is real-rooted.

Corollary: $(\mu_k)_{k=0}^{\lfloor n/2 \rfloor}$ forms an ultra log-concave sequence.

Proof: Note that $\mu_G(r)=0 \Rightarrow$

$$\mu_G(-r)=\sum_{k=0}^{\lfloor n/2\rfloor}\mu_k(-1)^k(-r)^{n-2k}$$

$$=(-1)^n\sum_{k=0}^{\lfloor n/2\rfloor}\mu_k(-1)^k r^{n-2k}=\mu_G(r)=0$$

$$\Rightarrow \mu_G(x)=\begin{cases}\prod_{i=1}^{n/2}(x^2-r_i), & n \text{ even}\\ x\prod_{i=1}^{\lfloor n/2\rfloor}(x^2-r_i), & n \text{ odd}\end{cases}$$

for some $r_i>0$ by Heilmann-Lieb.

Consider $f(x):=\prod_{i=1}^{\lfloor n/2\rfloor}(x+r_i)$ real-rooted.

$$\mu_G(x)\cong f(-x^2)$$

$$\Rightarrow f(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\mu_k x^{\lfloor n/2\rfloor-k}$$

$\Rightarrow \mu_k$ is ultra log-concave.

Proof of Heilmann-Lieb:

Mantra of Geom. of Polynomials:

More variables is better.

<u>Original proof</u>: Induction via deletion-contraction.

(Difficulty comes from the theory of sums of real-rooted polynomials: uses interlacing)

<u>Our proof</u>: More variables, two ways.

Index vertices by $\{1,2,\dots,n\}$.

<u>First</u>: Consider $M_G(x) = \left[\prod_{\{i,j\}\in E} (1-\partial x_i \partial x_j)\right] \prod_{i=1}^{n} x_i$

$= \left[\sum_{S\subseteq E} \prod_{\{i,j\}\in S} (-\partial x_i \partial x_j)\right] \prod_{i=1}^{n} x_i$

If $\{j,k\},\{j,l\}\in S$ for $k\neq l$, then

$\prod_{\{i,j\}\in S} (-\partial x_i \partial x_j) \prod_{i=1}^{n} x_i \equiv 0.$

Thus $= \left[\sum_{\substack{M\subseteq E \\ M \text{ matching}}} \prod_{\{i,j\}\in M} (-\partial x_i \partial x_j)\right] \prod_{i=1}^{n} x_i$

$$= \sum_{\substack{M \subseteq E, \\ M \text{ matching}}} (-1)^{\overset{\text{edge size}}{|M|}} x^{[n] \setminus M \leftarrow \text{vertices}}$$

$$M_G(t, \dots, t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \mu_k (-1)^k t^{n-2k} = \mu_G(t).$$

Thus, the theorem follows if $M_G(x)$ is real stable.

$$M_G(x) = \left[\prod_{\{i,j\} \in E} (1 - \partial_{x_i} \partial_{x_j}) \right] \prod_{i=1}^n x_i$$

Since $\prod_{i=1}^n x_i$ is real stable, follows if $1 - \partial_{x_i} \partial_{x_j}$ preserves stability on multiaffine polynomials.

Theorem for diff. ops.:

$$= (1 - \partial_{x_i} \partial_{x_j}) \prod_{k=1}^n x_k$$

$$= \underbrace{\left[\prod_{k \neq i,j} x_k \right]}_{\text{real stable}} \underbrace{(x_i x_j - 1)}_{\text{stable?}}$$

$x_i, x_j \in H_+ \Rightarrow$

Product of two elements of H_+ is in $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Thus $(x_i + z_i)(x_j + z_j) - 1 \neq 0$.

Therefore $(1 - \partial_{x_i} \partial_{x_j})$ preserves real stability, and thus $M_G(x)$ is real stable.

<u>Second</u>: (quickly)

$$x_1 \cdots x_n \cdot M_G(x^{-1}) = MAP\left[\prod_{\{i,j\} \in E} (1 - x_i x_j)\right],$$

where $MAP[x^\alpha] = \begin{cases} x^\alpha, & \bar{0} \leq \alpha \leq \bar{1} \\ 0, & \text{o.w.} \end{cases}$

"multiaffine part"

If MAP preserves real stability, then $M_G(x)$ is real stable

$$\text{Symb}^\lambda[MAP](x, z) = MAP\left[\prod_{i=1}^n (x_i + z_i)^{\lambda_i}\right]$$
$$= \prod_{i=1}^n \left(z_i^{\lambda_i} + \lambda_i z_i^{\lambda_i - 1} x_i\right)$$

$$= z^{\lambda - \bar{1}} \prod_{i=1}^{n} (z_i + \lambda_i x_i)$$

Since $\lambda_i \geq 0$, this is real stable.

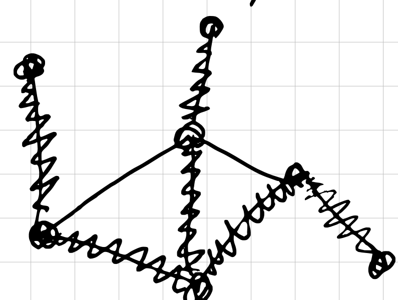
(Consider the geometry of polynomials "method" for this second proof: Encode graph as polynomial $\prod_{\{i,j\} \in E} (1 - x_i x_j)$, apply MAP, which preserves stability, extract info via $M_G(x_1, \dots, x_n) \longmapsto M_G(t, t, \dots, t)$.)

Lecture 6: Another application, and the link to matroids

(Another) example: The spanning tree generating polynomial.

Let $G=(V,E)$ be a connected graph. A <u>spanning tree</u> is a subgraph on the same vertex set which is a tree.

<u>E.g.</u>:



<u>Def</u>: The <u>spanning tree polynomial</u> of G is

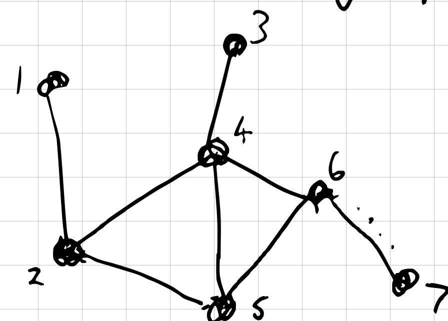
$$P_G(x) = \sum_{\substack{T \subseteq E, \\ \text{spanning tree}}} \prod_{e \in T} x_e.$$

(Now variables indexed by edges.)

Theorem (Kirchoff?):
For any connected graph G, $p_G(x)$ is real stable.

Proof:
Let E_G be the $|V| \times |E|$ edge-incidence graph of G

E.g.



$$E_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$E_G E_G^T = L_G$ = Laplacian of graph

Laplacian = diag(deg) − adjacency

Let E_G' be E_G w/ row 1 removed.

<u>Plan</u>:

1) $\sum_{T, \text{spanning tree}} x^T = \det(E_G' \operatorname{diag}(x) E_G'^T)$

via matrix tree thm. (Cauchy-Binet)

2) Show $\det(A \operatorname{diag}(x) A^T)$ is stable via linear algebra.

<u>Let's go</u>:

By Cauchy-Binet,

$$\det(E_G' \operatorname{diag}(x) E_G'^T)$$

$$= \sum_{S \in \binom{E}{|V|-1}} \det((E_G' \operatorname{diag}(x))_S) \det((E_G'^T)_S)$$

$$= \sum_{S \in \binom{E}{|V|-1}} x^S \det((E_G')_S (E_G')_S^T)$$

$$= \sum_{S \in \binom{E}{|V|-1}} x^S \cdot \det(E_S' E_S'^T)$$

<u>To show</u>: $\det(E_S' E_S'^T) = \delta_{\{S \text{ is spanning tree}\}}$

Because (V,S) has $|V|-1$ edges, either (V,S) is a (connected) tree, or else (V,S) contains a cycle.

If $C=\{e_1,e_2,\dots,e_k\}$ gives the edges of a cycle, then:

$$E_G' \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & & & 0 & 1 & \cdots \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & & 0 & \cdots \\ & 0 & & \ddots & 1 & \\ & & & & -1 \;\; -1 & \\ & \vdots & & & 0 \;\; 0 & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$$

Otherwise, (V,S) is a tree.

Let v_k be a leaf for some $k \geq 2$.

Then the $(k-1)^{th}$ row has exactly one non-zero entry, so by cofactor expansion,

$$|\det(E_S')| = |\det(E_{S \setminus v_k})|$$

remove vertex and edge

Since $|\det(E_G')| = 1$ when $G = \bullet\!\!-\!\!\bullet$, we have $|\det(E_S')| = 1$ by induction.

Thus, $\det(E_G' \operatorname{diag}(x) E_G'^T)$

$$= \sum_{T, \text{sp. tree}} x^T = p_G(x).$$

Now for (2).

For $a \in \mathbb{R}^E_{>0}$ and $b \in \mathbb{R}^E$, consider $p_G(a \cdot t + b)$

$$= \det(E_G'(A \cdot t + B) E_G'^T),$$

$$= \det((E_G' \cdot A \cdot E_G'^T) t + (E_G' B E_G'^T)),$$

where $A = \operatorname{diag}(a)$, $B = \operatorname{diag}(b)$.

If $p_G(x) \not\equiv 0$, then E_G' is full-rank, thus $E_G' \cdot A \cdot E_G'^T =: P$ is positive definite, and $E_G' \cdot B \cdot E_G'^T =: Q$ is real symmetric.

Thus, $p_G(a \cdot t + b) = \det(P) \cdot \det(t + P^{-1/2} Q P^{-1/2})$.

Since $P^{-1/2} Q P^{-1/2}$ is real symmetric, its eigenvalues are real, and thus

$$\det(x + P^{-1/2} Q P^{-1/2})$$

is real-rooted. Therefore, $p_G(x)$ is real stable. $\square$

<u>Corollary</u>: Consider the set of spanning forests with k components of G, $F_k(G)$. Then $p_{G,k}(x) = \sum_{S \in F_k(G)} p_S x^S$ is real stable, where p_S is the number of spanning trees which contain S.

<u>Proof</u>: Note that

$$p_{G,k}(x) = \Big(\sum_{R \in \binom{E}{k-1}} \partial_x^R\Big) p_G(x) = e_{k-1}((\partial_{x_i})_{i \in E})\, p_G(x).$$

So, if $e_k(\partial_1, \dots, \partial_n)$ preserves stability, then the result follows. Compute:

$$e_k(\partial_1, \dots, \partial_n)\, x^{(d, \dots, d)}$$
$$= d^k \cdot \underbrace{x^{(d-1, \dots, d-1)}}_{\hookrightarrow \text{stable}} \cdot \underbrace{e_{n-k}(x_1, \dots, x_n)}_{\hookrightarrow \text{stable?}}$$

$$e_{n-k}(x_1, \dots, x_n) = \mathrm{Pol}^n\left(\binom{n}{n-k} \cdot t^{n-k}\right).$$

Thus $e_k(\partial_1, \dots, \partial_n)$ preserves stability for polynomials of any degree. $\square$

(Now, by restricting to some line $a \cdot t + b$, we can obtain log-concavity inequalities for p_σ, $p_{\sigma,k}$.)

<u>Q</u>: Can we obtain inequalities on real stable polynomials like p_σ, $p_{\sigma,k}$ more directly?

More inequalities from real stable polynomials:
(For symmetric multiaffine real stable polynomials, ultra log-concavity implies inequalities between coefficients.)

<u>Theorem (Brändén, '07):</u>

For $p \in \mathbb{R}^{(1,1,\ldots,1)}[x_1,\ldots,x_n]$, p is real stable if and only if

$$R_{ij}(p) := \partial_{x_i} p \cdot \partial_{x_j} p - p \cdot \partial_{x_i}\partial_{x_j} p \geq 0$$

on $\mathbb{R}^n$.

(Crucial inequalities; we will see much more discussion when we study capacity)
(Only prove ($\Rightarrow$) direction)

<u>Proof:</u> ($\Rightarrow$) Real evaluation preserves real stability (limit upper half-plane evaluations and use Hurwitz theorem).

Thus we may assume $n=2$ by evaluating all other variables except i and j.

Now, $p \in \mathbb{R}^{(1,1)}[x_1, x_2] \to$

$$p(x_1, x_2) = a x_1 x_2 + b x_1 + c x_2 + d.$$

$$R_{12}(p) = \partial_{x_1} p \cdot \partial_{x_2} p - p \cdot \partial_{x_1} \partial_{x_2} p$$
$$= (a x_2 + b)(a x_1 + c) - (a x_1 x_2 + b x_1 + c x_2 + d) a$$
$$= bc - ad.$$

<u>Case 1</u>: $a=0$. To show: $b \cdot c \geq 0$

If $b < 0 < c$, let $x_1 = \frac{-d}{2b} + ci \in H_+$,

$$x_2 = \frac{-d}{2c} - bi \in H_+.$$

$$0 x_1 x_2 + b x_1 + c x_2 + d = bci - bci = 0$$

$\longrightarrow \Leftarrow$.

<u>Case 2</u>: $a \neq 0$. By dividing by a, we may assume $a=1$.

Now, $p(t-c, t-b) = (t^2 - (b+c)t + bc)$
$+ (bt - bc) + (ct - bc) + d = t^2 - (bc - d)$

Since this polynomial is real-rooted, we have that $bc - d \geq 0$ by discriminant. Since we assumed $a = 1$, this proves the claim. $\square$

<u>Example</u>: Spanning tree polynomial.

Fix connected G and two edges e, f. Consider:

$$0 \leq R_{ef}(p_G) = \partial_{x_e} p_G \cdot \partial_{x_f} p_G - p_G \partial_{x_e} \partial_{x_f} p_G$$

at all-ones vector.

$p_G(\bar{1})$ = # of span. trees

$\partial_{x_e} p_G(\bar{1})$ = " - - - - - - " cont. e

$\partial_{x_f} p_G(\bar{1})$ = " - - - - - - " cont. f

$\partial_{x_e} \partial_{x_f} p_G(\bar{1})$ = " - - - - - - " cont. e, f.

Divide by $p_G(\bar{1}) \cdot \partial_{x_f} p_G(\bar{1})$ to get:

$$0 \le \frac{\#e}{\text{total}} - \frac{\#e,f}{\#f}$$

$$\Rightarrow \mathbb{P}_T[e \in T \mid f \in T] \le \mathbb{P}_T[e \in T],$$

where T is unif. distributed.

<u>Negative dependence</u>:

Conditioning on an edge f being in T, the probability that e is in T <u>decreases</u>.

(Crucial to most applications of real stable/Lorentzian polynomials. Another way to phrase BB characterization: preservers of (strong) neg. dep.)

<u>Open question</u>: Similar theory for regular negative dependence?

Spanning forests polynomial.

Lecture 7 - Lorentzian polynomials

(Last time: Spanning tree polynomial and strong Rayleigh inequalities)

Recall: Theorem (Brändén, '07):

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$$R_{ij}(p) := \partial_{x_i} p \cdot \partial_{x_j} p - p \cdot \partial_{x_i} \partial_{x_j} p \geq 0$$

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$+ (bt - bc) + (ct - bc) + d = t^2 - (bc - d)$

Since this polynomial is real-rooted, we have that $bc - d \geq 0$ by discriminant. Since we assumed $a=1$, this proves the claim. $\square$

(Spanning tree polynomial is a generating polynomial for the bases of a graphic matroid. What about other matroids?)

<u>Lorentzian</u> polynomials try to "fill the gap" on two points related to real stable polynomials:

1) Real-rooted implies Newton's inequalities, but not converse.

2) The generating polynomial for graphic matroids are real stable, but not all matroids.

Definition: A d-homogeneous polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ is called **Lorentzian** (also strongly/completely log-concave) if:

(P) p has non-negative coeff.

(Q) $\forall v_1, \dots, v_{d-2} \in \mathbb{R}^n_{\geq 0}$, the Hessian of $D_{v_1} \cdots D_{v_{d-2}} p$ has at most one positive eigenvalue.

To "fill the gaps", we want to prove the following theorem:

Theorem (AOV '19, see also BH '19):

A d-homogeneous polynomial $p \in \mathbb{R}_{\geq 0}[x]$ is Lorentzian if and only if:

1) For all $\mu \in \mathbb{Z}^n_{\geq 0}$ with $|\mu| \leq d-2$, $\partial_x^\mu p$ is indecomposable

2) For all $\mu \in \mathbb{Z}^n_{\geq 0}$ with $|\mu| = d-2$, Hessian of $\partial_x^\mu p$ has at most one pos. e-val.

Def: A polynomial p is *indecomposable* if it cannot be written as $p = f + g$ where $f, g \not\equiv 0$ depend on disjoint sets of variables.

(Intuition: Comes from Perron–Frobenius theorem, related to connectivity of the associated graph.)

Q: How does this help?

Newton's Inequalities:

Fix $p \in \mathbb{R}_{\geq 0}[x, y]$ of degree d.

Theorem says p is Lorentzian iff

1) p has no "holes" in the coefficient sequence.

$$p = \cdots + p_k x^k y^{d-k} + p_{k+j} x^{k+j} y^{d-k-j} + \cdots$$

$$\mu = (k, d-k-j) \to \partial_x^k \partial_y^{d-k-j} = \tilde{p}_k y^j + \tilde{p}_{k+j} x^j.$$

2) Coefficient sequence of p satisfies Newton's inequalities.

$$p(x,y) = \sum_{j=0}^{d} \binom{d}{j} c_j x^j y^{d-j}$$

For any $0 \le k \le d-2$,

$$\partial_x^k \partial_y^{d-k-2} p = \binom{d}{k} k! \frac{(d-k)!}{2!} c_k y^2$$

$$+ \binom{d}{k+1} \frac{(k+1)!}{1!} \frac{(d-k-1)!}{1!} c_{k+1} xy + \binom{d}{k+2} \frac{(k+2)!}{2!} (d-k-2)! \, c_{k+2} x^2$$

$$\cong c_k y^2 + 2 c_{k+1} xy + c_{k+2} x^2$$

has at most one pos. eval.

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} c_{k+2} & c_{k+1} \\ c_{k+1} & c_k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = p(x,y)$$

$\Rightarrow M = \begin{bmatrix} c_{k+2} & c_{k+1} \\ c_{k+1} & c_k \end{bmatrix}$ has at most one pos. eval.

Since $c_{k+2}, c_k \ge 0$, matrix can't be negative definite. Thus,

$$c_k c_{k+2} - c_{k+1}^2 = \det(M) \le 0. \quad \square$$

Matroid basis generating polynomials

Definition: A *matroid* M on a finite ground set E is defined by a non-empty collection of subsets of E all of the same size (the *rank* of M), called *bases* of M, which satisfy:

(Exch) $\forall B_1, B_2 \in M$, $\forall i \in B_1 \setminus B_2$, $\exists j \in B_2 \setminus B_1$ such that
$$B_1 \cup \{j\} \setminus \{i\} \in M.$$

E.g.: The set of spanning trees of a conn. graph, where E is the set of edges of the graph.

Fact: If $M \setminus e$ is the set of bases in M which do not contain e, then $M \setminus e$ is either empty or a matroid. (This is the *deletion* of e.)

If M/e is the set given by removing e from all bases containing e, then M/e is either empty or a matroid. (This is the *contraction* of e.)

Proof: Exercise.

Let $p_M((x_e)_{e\in E}) := \sum_{B\in M} x^B$, called the *basis generating polynomial*.

Note that $\partial_{x_e} p_M = p_{M/e}$ and $p_M|_{x_e=0} = p_{M\setminus e}$.

Characterization theorem for p_M:

1) $\partial_x^S p_M$ indecomposable $\forall S \subseteq E$, $|S| \le d-2$

2) $\partial_x^S p_M$ (Hessian) at most one pos. e-val. $\forall S \subseteq E$, $|S| = d-2$.

Since $\partial_x^S p_M$ always gives $p_{M'}$ for some other matroid M', we can show that p_M is Lorentzian for all matroids if:

1) p_M is indecomposable for all M of rank ≥ 2.
2) p_M (Hessian) has at most one pos. eval. for all rank-2 M.

Corollary: For any matroid M, p_M is Lorentzian.

Proof: (1) Fix M and suppose

$$p_M(x) = f((x_e)_{e \in S}) + g((x_e)_{e \in S^c})$$

$f, g \not\equiv 0$.

Choose $B_1, B_2 \in M$ s.t. $B_1 \subseteq S$, $B_2 \subseteq S^c$. Apply exchange axiom to get $B := B_1 \cup \{f\} \setminus \{e\} \in M$, with $e \in S$, $f \in S^c$. Since rank ≥ 2, $B \cap S \neq \emptyset$, $B \cap S^c \neq \emptyset$, a contradiction.

(2) What does a rank-2 matroid look like?

By removing loops of M (unused vars.) we may assume every $e \in E$ is contained in some basis of M.

<u>Claim</u>: $\{e, f\} \notin M$ is an equivalence relation on E.

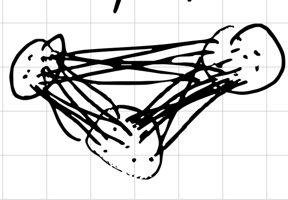
<u>Pf.</u>: $\not S$ $\{e_1, e_2\} \notin M$, $\{e_2, e_3\} \notin M$

If $\{e_1, e_3\} \in M$, fix any $\{e_2, f\} \in M$ and apply exch. axiom $\Rightarrow$

$\{e_2, f\} \cup \{e_i\} \setminus \{f\} \in M$ for some $i \in \{1, 3\}$ $\Rightarrow$ contradiction. $\square$

Thus, E breaks up into equiv. classes, and for any e, f in distinct classes, $\{e, f\} \in M$.

I.e., complete multipartite graph:



$G = (E, M)$

Now, $p_m(x) = x^T A x$, what does A look like? Order the variables by equivalence class:

$$
\begin{array}{c}
\begin{array}{c} C_1 \\ C_2 \\ \vdots \\ C_m \end{array}
\left[\begin{array}{c|c|c|c}
O & J & \cdots & J \\ \hline
J & O & \cdots & J \\ \hline
\vdots & \vdots & \ddots & \vdots \\ \hline
J & J & \cdots & O
\end{array}\right] \cdot \frac{1}{2} = A \\
\quad C_1 \quad C_2 \quad \cdots \quad C_m
\end{array}
$$

J = all-ones matrix

$$A = \frac{1}{2}\left(J_E - J_{C_1} - J_{C_2} - \cdots J_{C_m}\right)$$

$$= (\text{rank-one PSD}) - (\text{PSD})$$

Thus A has at most one positive eigenvalue. $\square$

Next: Prove the Lorentzian characterization theorem.

Proof strategy:

($\Rightarrow$) direction is easy.

($\Leftarrow$) direction by induction.

Base case is $d=2$ case, which is immediate.

Then by induction, $\partial_{x_i} p$ is Lorentzian. Need to use this to show that $\nabla_v p$ is Lorentzian for any $v \in \mathbb{R}^n_{\geq 0}$.