HW #1

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1 Interlacing polynomials and the BB characterization

Definition 1.1. Given a real-rooted polynomial $f \in \mathbb{R}[t]$, let $\lambda(f)$ denote the non-increasing sequence of the roots of f. Given $f, g \in \mathbb{R}[t]$ with positive leading coefficients, we say $f \ll g$ if one of the following holds:

- $\lambda_d(f) \leq \lambda_d(g) \leq \lambda_{d-1}(f) \leq \lambda_{d-1}(g) \leq \cdots \leq \lambda_1(f) \leq \lambda_1(g)$ with $\deg(f) = \deg(g) = d$
- $\lambda_d(g) \leq \lambda_{d-1}(f) \leq \lambda_{d-1}(g) \leq \cdots \leq \lambda_1(f) \leq \lambda_1(g)$ with $\deg(f) + 1 = \deg(g) = d$

We also say that f *interlaces* g. Further, we allow negative leading coefficients by saying that $f \ll g$ if and only if $g \ll -f$. Finally, we say that $f \ll g$ *strictly* (or f interlaces g *strictly*) if all inequalities above are strict.

1.1 Exercises

You may consult [Wag11] for hints on the exercises in this section, but try first to solve the problems yourself. Further, solutions must be written in your own words.

1. **Hermite-Biehler (HB) theorem.** Using the argument principle from complex analysis, prove the following. Given $f, g \in \mathbb{R}[t]$, the polynomial $g + if$ has all its roots in $\mathcal{H}_-$, the open lower half-plane, if and only if f, g are real-rooted with $f \ll g$ strictly. (**Hint:** When I say argument principle here I mean you should use the following. Suppose f is holomorphic in $\mathbb{C}$, C is a closed simple counterclockwise curve in $\mathbb{C}$, and f is never zero on C. Then the number of zeros of f (counting multiplicity) inside C is equal to the winding number of $f(C)$ about 0 (see https://en.wikipedia.org/wiki/Winding_number). That is, the number of zeros is equal to the (signed) number of times the closed curve $f(C)$ winds around 0. That said, consider a semicircular curve C given by $[-R, R] \cup \{R \cdot e^{i\theta} : \theta \in [-\pi, 0]\}$ for large R.)

2. **Hermite-Kakeya-Obreschkoff (HKO) theorem.** Given $f, g \in \mathbb{R}[t]$, prove that $af + bg$ is real-rooted for all $a, b \in \mathbb{R}$ if and only if f, g are real-rooted with either $f \ll g$ or $g \ll f$. (**Hint:** For the $\implies$ direction, consider the real solutions to the equation $\frac{f(t)}{g(t)} = c$ for any given $c \in \mathbb{R}$. For the $\impliedby$ direction, draw some plots of the two polynomials as functions.)

3. Use the HKO theorem to prove that if $V \subseteq \mathbb{R}[t]$ is a real linear subspace consisting only of real-rooted polynomials, then the dimension of V is at most two.

4. Use the previous exercise and the Hermite-Biehler theorem to prove that if $V \subseteq \mathbb{C}[t]$ is a complex linear subspace consisting only of stable polynomials, then the dimension of V is at most one.

5. Use the previous 2 exercises to complete the proofs of the complex and real BB characterizations in the univariate case. That is, show that if $T \in L(\mathbb{C}^d[x], \mathbb{C}[x])$ preserves stability and $\text{Symb}^d[T](x, z)$ is not stable, then the image of T is a complex linear space of stable polynomials of dimension at most one. Further, show that if $T \in L(\mathbb{R}^d[x], \mathbb{R}[x])$ preserves real-rootedness and neither $\text{Symb}^d[T](x, z)$

nor $\mathrm{Symb}^d[T](-x, z)$ is real stable, then the image of T is a real linear space of stable polynomials of dimension at most two. (**Hint:** Remember that we consider the zero polynomial to be both stable and real-rooted for these characterizations.)

6. Given multivariate $f, g \in \mathbb{R}[\boldsymbol{x}]$, we say $f \ll g$ whenever $f(\boldsymbol{a} \cdot t + \boldsymbol{b}) \ll g(\boldsymbol{a} \cdot t + \boldsymbol{b})$ for all $\boldsymbol{a} \in \mathbb{R}^n_{>0}$ and $\boldsymbol{b} \in \mathbb{R}^n$. Prove analogous results to the above exercises for the multivariate case, and use this to complete the proofs of the complex and real BB characterizations in the general case.

7. Given real-rooted $f, g, h \in \mathbb{R}[t]$ with positive leading coefficients such that $f \ll g$ and $f \ll h$, show that $ag + bh$ is real-rooted for all $a, b \geq 0$. Does this result extend to the case where f, g, h do not necessarily have positive leading coefficients?

8. Given $p \in \mathbb{C}[t]$ of degree d with all roots on the unit circle, prove that up to scalar the coefficients of p satisfy $p_k = \overline{p}_{d-k}$ for all k. What is the analogous result if "unit circle" is replaced by "real line", and how are these two results related? Prove versions of the HB and HKO theorems which hold for polynomials with roots on the unit circle.

2 Stability preservers

2.1 Exercises

1. Prove that if $p \in \mathbb{C}[\boldsymbol{x}]$ is homogeneous of degree d and stable, then all coefficients of p are non-negative up to complex scalar.

2. Define a homogenization operator on $\mathbb{R}^{\boldsymbol{\lambda}}[x_1, \ldots, x_n]$ via

$$T : \boldsymbol{x}^{\boldsymbol{\mu}} \mapsto \boldsymbol{x}^{\boldsymbol{\mu}} y^{|\boldsymbol{\lambda}| - |\boldsymbol{\mu}|},$$

   where $|\boldsymbol{\mu}| = \mu_1 + \cdots + \mu_n$ and y is a single variable. Use the BB characterization to prove that T does not preserve stability for any d, n.

3. Prove that if $p \in \mathbb{R}[\boldsymbol{x}]$ has non-negative coefficients, then its homogenization (of any fixed degree) is stable. (**Hint:** This is a bit of a challenge, use [Ren06] to prove it. Or find a proof that I don't know!) Note how this demonstrates a possible failing of the BB characterization: it cannot give more precise information about preservers for polynomials with non-negative coefficients.

4. Consider the partial derivative as an operator on spaces $\partial_t : \mathbb{R}^d[t] \to \mathbb{R}^{d-1}[t]$. Let $\omega_d : \mathbb{R}^d[t] \to \mathbb{R}^d[t]$ be the map given by $\omega_d(p) = t^d \cdot p(t^{-1})$. Recall the definition of $\tilde{\partial}_t : \mathbb{R}^d[t] \to \mathbb{R}^{d-1}[t]$, given by

$$\tilde{\partial}_t(p) = \omega_{d-1}[\partial_t(\omega_d[p])].$$

   Use the BB characterization to show that $\alpha\partial_t + \tilde{\partial}_t$ preserves stability for any $\alpha \in \mathcal{H}_+$, and that $a\partial_t + b\tilde{\partial}_t$ preserves real-rootedness for any $a, b \in \mathbb{R}$. Use this and some other exercises in this HW to demonstrate some relationship between the roots of $\partial_t p$ and $\tilde{\partial}_t p$ when p is real-rooted.

5. Given $p, q \in \mathbb{R}^d[t]$, define

$$[p \boxplus^d q](t) := \frac{1}{d!} \sum_{k=0}^{d} \partial_t^k p(x) \cdot \partial_t^{d-k} q(0).$$

   Prove that for any real-rooted $q \in \mathbb{R}^d[t]$, the linear operator $p \mapsto p \boxplus^d q$ preserves real-rootedness. Construct a linear differential operator (of the form $\sum_{k=0}^m c_k \partial_t^k$) which preserves real-rootedness on $\mathbb{R}^d[t]$ for some d, but doesn't preserve real-rootedness on $\mathbb{R}[t]$. Is there a family of linear differential operators which preserve real-rootedness on $\mathbb{R}^d[t]$ but not on $\mathbb{R}^{d+1}[t]$ for all d?

6. Let $P : \mathbb{R}^{(d,d)}[x, y] \to \mathbb{R}^d[t]$ be a linear operator defined on monomials via

$$P(x^j y^k) := t^j \boxplus^d t^k,$$

where $\boxplus^d$ is defined in the previous exercise. Prove or disprove: P maps real stable polynomials to real-rooted polynomials. How does this relate to the previous exercise?

7. Define a multivariate version $\boxplus^{\boldsymbol{\lambda}}$ of the operator considered in the previous two exercises, so that $\boxplus^{\boldsymbol{\lambda}}$ preserves real stability.

8. Given a graph $G = (V, E)$, a *k-factor* of G is a spanning subgraph of G in which all vertices have degree exactly k. Note that a 1-factor of a graph is a perfect matching. Let a weak k-factor be a subgraph of G in which all vertices have degree k or 0. Prove or disprove: There is a way to define real stable polynomials for k-factors or weak k-factors of a graph for some $k \geq 2$. (Note that you may need potentially to flip signs or something to make this work, or it may not work at all. This question is more open-ended.)

3 $\mathrm{SL}_2(\mathbb{C})$-action on polynomials

Let $\mathbb{C}^{\boldsymbol{\lambda}}[{\boldsymbol{x} \atop \boldsymbol{y}}] = \mathbb{C}^{\boldsymbol{\lambda}}[{x_1 \atop y_1}, \ldots, {x_n \atop y_n}]$ denote the linear space of polynomials which are homogeneous in (x_i, y_i) of degree λ_i for all $i \in [n]$ (including the zero polynomial). We will more succintly write these spaces of polynomials as $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$, where $\boldsymbol{v}_i \in \mathbb{C}^2$ for all $i \in [n]$, and we consider $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$ to have zeros in $(\mathbb{C}^2)^n$. We will not use this, but note that the homogeneity properties of p mean that the zeros of p can be considered to be elements of $(\mathbb{CP}^1)^n$, where $\mathbb{CP}^1$ is one-dimensional complex projective space (which is naturally isomorphic to the Riemann sphere). Let $H : \mathbb{C}^{\boldsymbol{\lambda}}[x_1, \ldots, x_n] \to \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$ denote the appropriate per-variable homogenization map.

Given $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n) \in \mathrm{SL}_2^n(\mathbb{C})$, we define an action of $\mathrm{SL}_2^n(\mathbb{C})$ on $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$ via:

$$\boldsymbol{\phi} \cdot p = p(\phi_1^{-1}\boldsymbol{v}_1, \ldots, \phi_n^{-1}\boldsymbol{v}_n),$$

where the $\boldsymbol{v}_i$ are considered to be column vectors. We also define $\boldsymbol{\phi} \cdot (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = (\phi_1\boldsymbol{v}_1, \ldots, \phi_n\boldsymbol{v}_n)$

3.1 Exercises

If you get stuck on the first few exercises, then restrict to the $n = 1$ (bivariate homogeneous) case. The general case follows straightforwardly from this case.

1. Given $p \in \mathbb{C}^{\boldsymbol{\lambda}}[x_1, \ldots, x_n]$, prove that p has a zero at $(\frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ if and only if $H(p)$ has a zero at $({r_1 \atop s_1}, \ldots, {r_n \atop s_n})$, including when $s_i = 0$ for some values of i.

2. Prove that $p \mapsto \boldsymbol{\phi} \cdot p$, as defined above, is a well-defined group action of $\mathrm{SL}_2^n(\mathbb{C})$ on $\mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$.

3. Given $p \in \mathbb{C}^{\boldsymbol{\lambda}}[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$ and $\boldsymbol{\phi} \in \mathrm{SL}_2^n(\mathbb{C})$, prove that $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ is a zero of p if and only if $\boldsymbol{\phi} \cdot (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ is a zero of $\boldsymbol{\phi} \cdot p$.

4. Given $p \in \mathbb{C}^d[{x \atop y}]$ and $\phi \in \mathrm{SL}_2(\mathbb{C})$, define $[{a \atop c}] = \phi^{-1}[{1 \atop 0}]$. We have

$$(\phi^{-1} \circ \partial_x \circ \phi) \cdot p = (a\partial_x + c\partial_y)p.$$

(Recall that we needed this lemma to prove Laguerre's theorem.)

5. Given polynomials $p, q \in \mathbb{C}^d[{x \atop y}]$, show that (up to non-zero scalar) the bilinear form of Grace's theorem can be given by

$$\langle H^{-1}(p), H^{-1}(q)\rangle = (\partial_x\partial_w - \partial_y\partial_z)^d p({x \atop y})q({z \atop w}).$$

Here we are identifying $\mathbb{C}^d[x]$ and $\mathbb{C}^d[{x \atop y}]$ via the map H.

6. **Multivariate Grace's theorem.** Given polynomials $p, q \in \mathbb{C}^{\boldsymbol{\lambda}}[\begin{smallmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{smallmatrix}] = \mathbb{C}^{\boldsymbol{\lambda}}[\begin{smallmatrix}x_1\\ y_1\end{smallmatrix}, \ldots, \begin{smallmatrix}x_n\\ y_n\end{smallmatrix}]$, prove that if $H^{-1}(p)$ is $\mathcal{H}_+^n$-stable and $H^{-1}(q)$ is $\overline{\mathcal{H}_-}^n$-stable, then

$$\langle H^{-1}(p), H^{-1}(q)\rangle := \left[\prod_{i=1}^{n}(\partial_{x_i}\partial_{w_i} - \partial_{y_i}\partial_{z_i})^{\lambda_i}\right] p(\begin{smallmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{smallmatrix})q(\begin{smallmatrix}\boldsymbol{z}\\ \boldsymbol{w}\end{smallmatrix}) \neq 0.$$

(**Hint:** This is a harder problem. The quickest way is likely to show that $\partial_x\partial_w - \partial_y\partial_z$ preserves $(\mathcal{H}_+ \times \overline{\mathcal{H}_-})$-stability on $\mathbb{C}^{(d,d)}[\begin{smallmatrix}x\\ y\end{smallmatrix}, \begin{smallmatrix}z\\ w\end{smallmatrix}]$, *not* allowing the zero polynomial. However, we have not really discussed too much in the course how to disallow the zero polynomial. See [Lea17] for more discussion on this problem.)

7. Prove that $\partial_x\partial_w - \partial_y\partial_z$ is $\mathrm{SL}_2(\mathbb{C})$-invariant. That is, given $\phi \in \mathrm{SL}_2(\mathbb{C})$ and $f \in \mathbb{C}^{(d,d')}[\begin{smallmatrix}x\\ y\end{smallmatrix}, \begin{smallmatrix}z\\ w\end{smallmatrix}]$, we have

$$(\phi, \phi) \cdot [(\partial_x\partial_w - \partial_y\partial_z)f] = (\partial_x\partial_w - \partial_y\partial_z)[(\phi, \phi) \cdot f].$$

Note that this implies the multivariate Grace's theorem of the previous exercise generalizes to any product of circular regions and the product of their complements.

8. Prove or disprove: The action of $\mathrm{SL}_2(\mathbb{R})$ preserves the interlacing property on univariate polynomials. That is, if $f \ll g$ then $\phi \cdot f \ll \phi \cdot g$ for all $\phi \in \mathrm{SL}_2(\mathbb{R})$. Here we are implicitly identifying $\mathbb{R}^d[x]$ and $\mathbb{R}^d[\begin{smallmatrix}x\\ y\end{smallmatrix}]$ via the map H. (**Hint:** What makes this interesting is what happens to the sign of the leading coefficient when the action of some ϕ causes roots to "pass through infinity".)

References

[Lea17] Jonathan Leake, *A representation theoretic explanation of the Borcea-Brändén characterization*, arXiv preprint arXiv:1706.06168 (2017).

[Ren06] James Renegar, *Hyperbolic programs, and their derivative relaxations*, Foundations of Computational Mathematics **6** (2006), no. 1, 59–79.

[Wag11] David Wagner, *Multivariate stable polynomials: Theory and applications*, Bulletin of the American Mathematical Society **48** (2011), no. 1, 53–84.